Homework 9

The homeworks are due on the Thursday of the week after the assignment was posted online[1]. Please hand in your homework at the beginning of the tutorial or bring it to the lecture on Thursday morning. You can work on and submit your homework in groups of two. Please staple your pages and write your names and matriculation numbers on the first page.

Problem 25 (10 pts.)

a) If L/K is Galois and w is an extension of v to L, then $w \circ \sigma$ is an extension of v for all $\sigma \in \mathrm{Gal}(L/K)$.

b) Let L/K be a finite Galois extension and let K be complete with respect to the exponential valuation v. Give a new proof (different from the one given in class) that $w(x) = \frac{1}{n}v(N_{L/K}(x))$ is the unique extension of v to L.
Hint: You can use, as seen in class, that there is at least one extension of v to L, and you should use part a) to prove b).

Solution. a) This is a trivial calculation: First, since σ acts trivially on K, $w(\sigma(x)) = w(x) = v(x)$ for all $x \in K$. Thus, if $w \circ \sigma$ is a valuation on L, then it is an extension of v to L. Now $w(\sigma(x)) = 0$ is equivalent to $\sigma(x) = 0$ and this happens if and only if $x = 0$ since σ is bijective. Moreover, $w(\sigma(xy)) = w(\sigma(x)\sigma(y)) = w(\sigma(x)) + w(\sigma(y))$ because σ is a homomorphism and w a valuation. Finally,

$$w(\sigma(x+y)) = w(\sigma(x) + \sigma(y)) \geq \min\{w(\sigma(x)), w(\sigma(y))\}$$

since σ is a homomorphism and w is a valuation.
b) We know there is at least one extension w of v to L. Let w be any such extension and let $\sigma \in \mathrm{Gal}(L/K)$. By Satz 2.3 in class and a) we then conclude that w and $w \circ \sigma$ are equivalent (as norms and hence as valuations). Since they agree on K (and are both valuations!), it follows that $w = w \circ \sigma$. Consequently,

$$w(x) = \frac{1}{n} \sum_{\sigma \in \mathrm{Gal}(L/K)} w(\sigma(x)) = \frac{1}{n} w\left(\prod_{\sigma \in \mathrm{Gal}(L/K)} \sigma(x) \right) = \frac{1}{n} v\left(\mathrm{N}_{L/K}(x)\right).$$

Problem 26 (10 pts.)

Let K be a local field with discrete valuation v. Let $f \in K[X]$ given by

$$f(X) = a_n X^n + \cdots + a_1 X + a_0$$

[1] This assignment is due Thursday, 12.12.19.

with $a_0 a_n \neq 0$.
First look up the definition of the Newton polygon of f from Exercise 27 below.
Let $L = K(\alpha_1, \ldots, \alpha_n)$ be the splitting field of f over K. We also write v for the unique extension of the valuation from K to L. Let us suppose that $\alpha_1, \ldots, \alpha_n$ are ordered, such that

$$\begin{aligned} v(\alpha_1) = \cdots = v(\alpha_{s_1}) &= m_1 \\ v(\alpha_{s_1+1}) = \cdots = v(\alpha_{s_2}) &= m_2 \\ \cdots &= \cdots \\ v(\alpha_{s_{r-1}+1}) = \cdots = v(\alpha_n) &= m_r \end{aligned}$$

with $m_1 < m_2 < \ldots < m_r$.
In the following, we suppose that $a_n = 1$ for simplicity. This is not a restriction, as the main point c) remains valid if $a_n \neq 1$ (because multiplication with a constant might only shift the Newton polygon up or down).

a) Show that

$$\begin{aligned} v(a_n) &= 0 \\ v(a_{n-1}) &\geq m_1 \\ v(a_{n-2}) &\geq 2m_1 \\ &\cdots \\ v(a_{n-s_1}) &= s_1 m_1 \\ v(a_{n-s_1-1}) &\geq s_1 m_1 + m_2 \\ &\cdots \\ v(a_{n-s_2}) &= s_1 m_1 + (s_2 - s_1) m_2 \\ &\cdots \\ v(a_{n-s_2-1}) &\geq s_1 m_1 + (s_2 - s_1) m_2 + m_3 \\ &\cdots \\ v(a_0) &= s_1 m_1 + (s_2 - s_1) m_2 + (s_3 - s_2) m_3 + \cdots + (n - s_{r-1}) m_r. \end{aligned}$$

Hint: Look up formulas for the coefficients a_i in terms of elementary symmetric polynomials in the roots α_j in an algebra book.

b) Show that the vertices of the Newton polygon are (from right to left) exactly the points

$$(n, 0), (n - s_1, s_1 m_1), (n - s_2, s_1 m_1 + (s_2 - s_1) m_1), \ldots,$$

and the slopes of the lines connecting these vertices are exactly $-m_1, -m_2, \ldots, -m_r$.

c) Summarize: If $(k, v(a_k)) \leftrightarrow (\ell, v(a_\ell))$ is a line ($k < \ell$) of the Newton polygon with slope $-m$, then f has exactly $\ell - k$ roots with valuation m.

Solution. a) By expressing the coefficients a_i in terms of the roots α_i we obtain

$$\begin{aligned} v(a_n) &= v(1) = 0, \\ v(a_{n-1}) &\geq \min_i \{v(\alpha_i)\} = m_1, \\ v(a_{n-2}) &\geq \min_{i,j} \{v(\alpha_i \alpha_j)\} = 2m_1, \\ &\cdots \\ v(a_{n-s_1}) &= \min_{i_1, \ldots, i_{s_1}} \{v(\alpha_{i_1} \ldots \alpha_{i_{s_1}})\} = s_1 m_1, \end{aligned}$$

the latter being an equality because $v(\alpha_1 \dots \alpha_s) = s_1 m_1$ is smaller than all the others,

$$\begin{aligned} v(a_{n-s_1-1}) &\geq \min_{i_1,\dots,i_{s_1+1}} \{v(\alpha_{i_1} \dots \alpha_{i_{s_1+1}})\} = s_1 m_1 + m_2, \\ v(a_{n-s_1-2}) &\geq \min_{i_1,\dots,i_{s_1+2}} \{v(\alpha_{i_1} \dots \alpha_{i_{s_1+2}})\} = s_1 m_1 + 2m_2, \\ &\dots \\ v(a_{n-s_2}) &= \min_{i_1,\dots,i_{s_2}} \{v(\alpha_{i_1} \dots \alpha_{i_{s_2}})\} = s_1 m_1 + (s_2 - s_1) m_2, \end{aligned}$$

and so on. b) From the previous part we conclude that the vertices of the Newton polygon, from right to left, are given by

$$(n, 0), \quad (n - s_1, s_1 m_1), \quad (n - s_2, s_1 m_1 + (s_2 - s_1) m_2), \quad \dots$$

The slope of the most right-hand point will be given by

$$\frac{0 - s_1 m_1}{n - (n - s_1)} = -m_1,$$

and, moving to the left, the slopes will be given by

$$\frac{(s_1 m_1 + \dots + (s_j - s_{j-1}) m_j) - (s_1 m_1 + \dots + (s_{j+1} - s_j) m_{j+1})}{(n - s_j) - (n - s_{j+1})} = -m_{j+1},$$

and part c) easily follows. □

Problem 27 (10 pts.)

We use the same notation as in Problem 26 and suppose that K is a local field of characteristic 0. Show that there is a decomposition

$$f(X) = a_n \prod_{j=1}^{r} f_j(X)$$

with

$$f_j(X) = \prod_{i=s_{j-1}+1}^{s_j} (X - \alpha_i) \in K[X]$$

and with $s_0 = 0$. In particular: If the Newton polygon has n different slopes, then f splits completely over K. If the polynomial f is irreducible, then the Newton polygon has only one slope.

Hints:

- Recall that if L/K is the splitting field of f, then L/K is a Galois extension.
- First suppose that f is irreducible.
- Then give a proof by induction on the degree n.

Solution. What we need to prove is that if the valuation v on K admits a unique extension w to the splitting field L of f over K, then the factorization

$$f(X) = a_n \prod_{j=1}^{r} f_j(X)$$

is already defined over K, that is, $f_j(X) = \prod_{w(\alpha_i)=m_j}(X-\alpha_i) \in K[X]$.
The statement is clear when f is irreducible because then one has $\alpha_i = \sigma_i\alpha_1$ for some $\sigma_i \in \mathrm{Gal}(L/K)$ and, since $v \circ \sigma_i$ is again a valuation, the uniqueness implies that $v(\alpha_i) = v(\sigma_i\alpha_1) = m_1$, hence $f_1(x) = f(x)$.
The general case follows by induction on n. If $n = 1$, there is nothing to prove. Let p be the minimal polynomial of α_1 and $g(x) = f(x)/p(x) \in K[X]$. Since all roots of p have the same valuation m_1, p must be a divisor of f_1. Let $g_1(x) = f_1(x)/p(x)$. The factorization of $g(x)$ according to the slopes is

$$g(x) = g_1(x)\prod_{j=2}^{r} f_j(x).$$

Since $\deg(g) < \deg(f)$, it follows that $f_j(x) \in K[x]$ for all $= 1, \ldots, r$. □

Bonus Problem (10 pts.)

Let K be a finite extension of $\mathbb{Q}_p$ of degree n. Let $\mathcal{O}_K$ be its valuation ring, $\mathfrak{p}$ its maximal ideal and $\mathbb{k} = \mathcal{O}_K/\mathfrak{p}$ its residue field.
Let e be the *ramification index* defined in Homework 8 and let f be the *degree of inertia*, i.e., the degree of the extension of finite fields $[\mathbb{k} : \mathbb{F}_p]$.
Prove the *characteristic equation*:

$$n = e \cdot f.$$

(This shows that the degree $n = [K : \mathbb{Q}_p]$ of a finite extension K of $\mathbb{Q}_p$ breaks up as a product $n = e \cdot f$, where e "measures" the change of the image of the p-adic valuation ν_p and $f = [\mathbb{k} : \mathbb{F}_p]$ the change in the residue field.)

Solution. This is Proposition 5.4.6 in Gouvêa. □

The following exercises will be discussed in the tutorial and you do not need to hand in solutions for them.

Exercise 27

Newton polygons.
Let K be a local field with discrete valuation v. Let $f \in K[X]$ given by

$$f(X) = a_nX^n + \cdots + a_1X + a_0$$

with $a_0a_n \neq 0$. Consider the set of points $S = \{(i, v(a_i)) \in \mathbb{R}^2 : i = 0, \ldots, n, a_i \neq 0\}$. The Newton polygon of f is then defined to be the lower boundary of the convex hull of S.
Draw the Newton polygon of each of the following two polynomials, compute the slopes and illustrate Problem 26, c) in these cases.

1. $f(X) = 5^6X^4 + 5^3X^3 - 5X^2 - X + \frac{1}{7} \in \mathbb{Q}_5[X]$
2. $f(X) = 5^2X^5 - \sqrt{5}X^3 + 7X^2 - 1 \in \mathbb{Q}_5(\sqrt{5})[X]$