Homework 8

The homeworks are due on the Thursday of the week after the assignment was posted online[1]. Please hand in your homework at the beginning of the tutorial or bring it to the lecture on Thursday morning. You can work on and submit your homework in groups of two. Please staple your pages and write your names and matriculation numbers on the first page.

Problem 22 (10 pts.)

Let p be a prime number. Let $\overline{\mathbb{Q}}_p$ be the algebraic closure of $\mathbb{Q}_p$, i.e., the (up to isomorphism) unique algebraic extension K of $\mathbb{Q}_p$ that is algebraically closed. We can construct $\mathbb{Q}_p$ as the splitting field of all polynomials in $\mathbb{Q}_p[X]$ or, equivalently, the union of all finite extensions of $\mathbb{Q}_p$ (and then show that this is an algebraically closed field). We know that the algebraic closure of $\mathbb{R}$ is $\mathbb{C}$ and thus $\overline{\mathbb{R}}$ has degree 2 over $\mathbb{R}$.
Show that, however, the extension $\overline{\mathbb{Q}}_p/\mathbb{Q}_p$ is infinite.
Hint 1: Show that, for every $n \in \mathbb{N}$, there is an irreducible polynomial $f \in \mathbb{Q}_p[X]$ of degree n. For this you may use, for example, the reduction criterion for polynomials over a unique factorization domain and Hint 2.
Hint 2: Use the following fact from Algebra. For all $n \in \mathbb{N}$ there is a finite field $\mathbb{F}_{p^n}$ with p^n elements. The extension $\mathbb{F}_{p^n}/\mathbb{F}_p$ is a Galois extension of degree n.

Problem 23 (10 pts.)

Let $K/\mathbb{Q}_p$ be a finite extension, $|\ |$ the unique extension of the p-adic absolute value to K. Let $\mathcal{O}_K \subset K$ be the valuation ring. Show that $\mathcal{O}_K$ consists exactly of those elements $x \in K$ that are roots of a monic polynomial with coefficients in $\mathbb{Z}_p$ (i.e., $\mathcal{O}_K$ is the "ring of integers" in K).

Problem 24 (10 pts.)

Let $e = e(K/\mathbb{Q}_p)$ be the *ramification index* as defined in Exercise 26. If we write $f = f(K/\mathbb{Q}_p) = n/e$ and if $\Bbbk = \mathcal{O}/\mathfrak{p}$ denotes the residue field, then $[\Bbbk : \mathbb{F}_p] = f$, so that $\Bbbk = \mathbb{F}_{p^f}$ is the finite field with p^f elements. You don't need to prove this! Instead, just try to exemplify this result by computing e, f and n for the following fields:

(i) $p = 5,\ K = \mathbb{Q}_5(\sqrt{2})$.

(ii) $p = 5,\ K = \mathbb{Q}_5(\sqrt{5})$.

(iii) $p = 5, K = \mathbb{Q}_5(\sqrt{11})$.

(iv) **Bonus (5 pts.)**: $p = 3,\ K = \mathbb{Q}_3(\zeta, \sqrt{2})$, where ζ is a primitive third root of unity, i.e., $\zeta^3 = 1$ but $\zeta \neq 1$.

[1] This assignment is due Thursday, 05.12.19.

The following exercises will be discussed in the tutorial and you do not need to hand in solutions for them.

Exercise 25

Let $p = 5$. Check that 2 is not a square in $\mathbb{Q}_5$ and consider the quadratic extension $K = \mathbb{Q}_5(\sqrt{2})$. View K as a 2-dimensional $\mathbb{Q}_5$-vector space and give an example of a (vector space) norm on K that is not a valuation on K but does extend the p-adic absolute value on $\mathbb{Q}_5$.

Exercise 26

Let $K/\mathbb{Q}_p$ be a finite extension of degree n. Let v be the unique extension of the p-adic valuation v_p to K.

(a) Show that there is a positive integer $e \geq 1$, such that $e \mid n$ and

$$v(K^\times) = \frac{1}{e}\mathbb{Z}.$$

This integer is called the *ramification index* of $K/\mathbb{Q}_p$.

(b) Let $K = \mathbb{Q}_5(\sqrt{2})$, which is a quadratic extension of $\mathbb{Q}_5$. Find the ramification index $e = e(K/\mathbb{Q}_5)$ in this example.