Homework 7

The homeworks are due on the Thursday of the week after the assignment was posted online[1]. Please hand in your homework at the beginning of the tutorial or bring it to the lecture on Thursday morning. You can work on and submit your homework in groups of two. Please staple your pages and write your names and matriculation numbers on the first page.

Problem 19 (10 pts.)

Show that $|z| = (z\overline{z})^{1/2} = \sqrt{\mathrm{N}_{\mathbb{C}/\mathbb{R}}(z)}$ is the only valuation of $\mathbb{C}$ which extends the absolute value $|\ |$ of $\mathbb{R}$.

Solution. To check that $|\ |$ is a valuation is easy.
We need to show uniqueness. To prove this, suppose $|\ |$ and $\|\ \|$ are two valuations on $\mathbb{C}$ which extend the usual absolute value from $\mathbb{R}$. We view $\mathbb{C}$ as a 2-dimensional $\mathbb{R}$-vector space and we recall from analysis that any two norms on a finite dimensional $\mathbb{R}$-vector space are equivalent. This means that $|\ |$ and $\|\ \|$ induce the same topology on $\mathbb{C}$ and hence there is an $s \in \mathbb{R}$, such that $|x| = \|x\|^s$.
However, since $|x| = \|x\|$ for all $x \in \mathbb{R}$, we must have $s = 1$ and we are done. $\square$

Problem 20 (10 pts.)

Let k be a field and $K = k(t)$ the function field in one variable. Show that the valuations $\nu_{\mathfrak{p}}$ associated to the prime ideals $\mathfrak{p} = (p(t))$ of $k[t]$, together with the degree valuation ν_∞, are the only (up to equivalence) non-trivial valuations of K that are trivial on k. What are the corresponding valuation rings, maximal ideals and residue fields?
Hints: Start by using Exercise 22 to clarify the basics. Then make a case distinction according to whether a valuation ν satisfies $\nu(f) \geq 0$ for all $f \in k[t]$, or whether there exists $f \in k[t]$ with $\nu(f) < 0$.

Solution. Suppose first that $\nu(f) \geq 0$ for any $f \in k[t]$. Let $p \in k[t]$ be of smallest degree such that $\nu(f) > 0$. Clearly p must be an irreducible polynomial. Consider $\mathfrak{p} = \{f \in k[t] : \nu(f) > 0\}$, which is an ideal (and which is not the whole K, since $\nu = 0$ on $K^\times$ by assumption). It is easy to see that $(p) \subseteq \mathfrak{p}$. But (p) is also a maximal ideal, so it must be that $\mathfrak{p} = (p)$ and by Exercise 22 we conclude that ν is just a multiple of $\nu_{\mathfrak{p}}$. The valuation ring is $\mathcal{O} = k[t]_{(p)} = \{f = \frac{g}{h} \in K : \ p \nmid h\}$, its maximal ideal is $\mathfrak{P} = p \cdot k[t]_{(p)}$ and its residue field is $\mathcal{O}/\mathfrak{P} = k[t]_{(p)}/pk[t]_{(p)} \cong k[t]/pk[t]$, which is a finite field extension of k of degree $n = \deg(p)$, in which p has a root.
If there exists $f \in k[t]$ with $\nu(f) < 0$, then it follows also that $\nu(t) < 0$. WLOG (say, by scaling), we can assume $\nu(t) = -1$, in which case $\nu(t^n) = -n$. It follows by the strong triangle inequality (ultrametric identity) that $\nu\big(\sum_{k=0}^{N} a_k t^k\big) = -N$, and we notice that this gives us the degree valuation ν_∞ as defined in Exercise 22. The valuation ring is $\mathcal{O} = \{f/g \in K : \deg(f) \leq \deg(g)\}$, its maximal ideal is $\mathfrak{P} = (t^{-1})$ and its residue field is $\mathcal{O}/\mathfrak{P} = k$. $\square$

[1] This assignment is due Thursday, 28.11.19.

Problem 21 (10 pts.)

Krull valuations. Let $\mathcal{O}$ be an arbitrary valuation ring with field of fractions K, and let $\Gamma = K^*/\mathcal{O}^*$. Prove that Γ is a totally ordered group if we define $x \pmod{\mathcal{O}^*} \geq y \pmod{\mathcal{O}^*}$ to mean $x/y \in \mathcal{O}$. Write Γ additively and show that the function (called *Krull valuation*)

$$\nu : K \to \Gamma \cup \{\infty\}$$

defined by $\nu(0) = \infty$, $\nu(x) = x \pmod{\mathcal{O}^*}$ for $x \in K^*$, satisfies the conditions

(1) if $\nu(x) = \infty$, then $x = 0$.

(2) $\nu(xy) = \nu(x) + \nu(y)$.

(3) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$.

Solution. The fact that Γ is totally ordered with respect to $\geq$ follows from the fact that, since $\mathcal{O}$ is a valuation ring, either $x/y \in \mathcal{O}$ or $y/x \in \mathcal{O}$. Further, the fact that $x \geq y$ and $y \geq z \implies x \geq z$ and the fact that $x \geq y \implies xz \geq yz$ (for any z) follow both immediately from the definition. Properties (1) and (2) are clearly satisfied, so let us prove (3). Take any $x, y \in K^*$ and assume $\nu(x) \leq \nu(y)$. This means that $yx^{-1} \in \mathcal{O}$. Since $\mathcal{O}$ is a commutative ring with identity, $1 + yx^{-1} \in \mathcal{O}$, hence $1 + yx^{-1} = (1 + yx^{-1})/1 \in \mathcal{O}$, and so $1 \pmod{\mathcal{O}^*} \leq 1 + yx^{-1} \pmod{\mathcal{O}^*}$. Setting $x = y = 1$ in (2) gives $\nu(1) = 2\nu(1)$, thus $\nu(1) = 0$ (which was to be expected anyway, as the multiplicative identity 1 of K should get sent to the additive identity 0 of Γ). We then have

$$\begin{aligned} \nu(x + y) &= \nu(x(1 + yx^{-1})) = \nu(x) + \nu(1 + yx^{-1}) \\ &\geq \nu(x) + \nu(1) \\ &= \nu(x) = \min\{\nu(x), \nu(y)\}. \end{aligned}$$

□

The following exercises will be discussed in the tutorial and you do not need to hand in solutions for them.

Exercise 21

Is the polynomial ring over a field a valuation ring (in its field of fractions)?

Solution. No, as for any relatively prime or irreducible polynomials $f, g \in k[X]$, neither f/g nor g/f is in $k[X]$. □

Exercise 22

In this exercise we prepare Problem 20. Let k be a field and $K = k(t)$ the function field in one variable over k. Recall that the polynomial ring $R = k[t] \subset K$ is a principal ideal domain with field of fractions K.

(i) Let $\mathfrak{p} = (p(t))$ be a prime ideal in R. Show that there is an associated (exponential) valuation $v_{\mathfrak{p}} : K \to \mathbb{Z} \cup \{\infty\}$.

(ii) Define an exponential valuation $\nu_\infty : K \to \mathbb{Z} \cup \{\infty\}$ using the degree map. **Hint:** Start on R and check that $\nu_\infty(f) = -\deg(f)$ satisties the required properties. Then extend this to K.

Solution. (i) We know that R is a PID (principal ideal domain), hence a UFD (unique factorization domain). If $\mathfrak{p} = (p(t))$ is a prime ideal, i.e., $p(t)$ is irreducible, we can write any $f \in K \setminus \{0\}$ as $f = p^{\nu}\frac{g}{h}$, with $\nu \in \mathbb{Z}$ and $\gcd(g,h) = \gcd(p,g) = \gcd(p,h) = 1$; define $\nu_{\mathfrak{p}}(f) = \nu$ and $\nu_{\mathfrak{p}}(0) = \infty$. (ii) Set again $\nu_\infty(0) = \infty$ and, for any $f/g \in K \setminus \{0\}$, let $\nu_\infty(f/g) = \deg(g) - \deg(f)$. □

Exercise 23

Determine the valuation ring $\mathcal{O}$ in $\mathbb{Q}$, its maximal ideal $\mathfrak{P}$ and its residue field with respect to the p-adic exponential valuation ν_p. What is the relation between the ring $\mathcal{O}$ and $\mathbb{Z}_p$?

Solution. The valuation ring is $\mathcal{O} = \mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} : p \nmid b\}$ and its valuation ideal is $\mathfrak{P} = p\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} : p \nmid b \text{ and } p \mid a\}$. We have $\mathbb{Z}_{(p)} = \mathbb{Q} \cap \mathbb{Z}_p$. The residue field is $\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} \cong \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$. For the last claim, note that there is an injective homomorphism $\mathbb{Z} \hookrightarrow \mathbb{Z}_{(p)}$ which maps $p\mathbb{Z}$ into $p\mathbb{Z}_{(p)}$. This gives an injective map $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$ (because $a \in \mathbb{Z}$ maps to 0 only if $a/1 \in p\mathbb{Z}_{(p)}$). To see this map is also onto, note that if $p \nmid b$, then there exists b_1 such that $bb_1 \equiv 1 \pmod{p}$, in which case, for any $a/b \in \mathbb{Z}_{(p)}$, the integer ab_1 maps to the class of a/b in $\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$. □

Exercise 24

Let k be a field and $k((T))$ the field of Laurent power series in the variable T. If $f = \sum_{n \geq n_0} a_n T^n$ with $a_{n_0} \neq 0$, define the order of f as $\nu(f) = n_0$. This defines an (exponential) valuation. Determine the valuation ring and show that it is a discrete valuation ring. What is its residue field?

Solution. The valuation ring is the ring $\mathcal{O} = k[[T]]$ of formal power series and is clearly a discrete valuation ring. To find the residue field, we need to find the maximal ideal of $\mathcal{O}$, that is, $\mathfrak{P} = \{f \in \mathcal{O} : v(f) \geq 1\} = (T)$, which gives $\mathcal{O}/\mathfrak{P} = k$. □