Homework 6

The homeworks are due on the Thursday of the week after the assignment was posted online[1]. Please hand in your homework at the beginning of the tutorial or bring it to the lecture on Thursday morning. You can work on and submit your homework in groups of two. Please staple your pages and write your names and matriculation numbers on the first page.

Problem 16 (10 pts.)

(i) Use power series to define p-adic analogues "$\sin_p$" and "$\cos_p$" of the usual sin and cos functions, and determine their regions of convergence.

(ii) Show that if $p \equiv 1 \pmod 4$, then there exists $i \in \mathbb{Q}_p$ such that $i^2 = -1$ and the classical relation

$$\exp_p(ix) = \cos_p(x) + i \sin_p(x)$$

holds for any x in the convergence region.

(iii) We know that the classical trigonometric functions sin and cos are periodic. Are the p-adic analogues also periodic?

Solution. (i) We define, in a natural way,

$$\sin_p(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{and} \quad \cos_p(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots ,$$

which will have the same region of convergence as the exponential function $\exp_p$, namely the open ball of radius $p^{-1/(p-1)}$. (ii) By Hensel's lemma, it is enough to find a solution of the equation $x^2 + 1 \equiv 0 \pmod p$, and this follows easily by computing the Legendre symbol $(-1/p) = (-1)^{(p-1)/2} = 1$ for $p \equiv 1 \pmod 4$. The formula is clear by comparing the coefficients. (iii) No. □

Problem 17 (10 pts.)

Let $X \subset \mathbb{Q}_p$ be a compact and open subset. Show that a function $f : X \to \mathbb{Q}_p$ is locally constant if and only if there are finitely many compact and open subsets $U_1, \ldots, U_n \subset X$ such that

$$f = \sum_{i=1}^{n} a_i \mathbf{1}_{U_i}$$

with $a_i \in \mathbb{Q}_p$. Here, $\mathbf{1}_{U_i} : X \to \{0, 1\}$ is the characteristic function of U_i, i.e $\mathbf{1}_{U_i}(x) = 1$ if $x \in U_i$ and $\mathbf{1}_{U_i}(x) = 0$ otherwise.

[1]This assignment is due Thursday, 21.11.19.

Solution. "$\Rightarrow$" Around every point $x \in X$ one can find an open (and closed, hence compact) ball U_x of radius p^{-m}, i.e., $U_x = \{y \in \mathbb{Z}_p : |x-y|_p < p^{-m}\}$, for some $m \in \mathbb{Z}$, on which f is constant. Now, since $\bigcup_{x \in X} U_x$ is an open cover of X and X is compact, we can extract a *finite* subcover $\bigcup_{i=1}^n U_{x_i}$ of X. On each such ball f will be constant, say $f(x) = a_i \in \mathbb{Q}_p$ for $x \in U_{x_i}$. Any two such balls will either be disjoint, or one will contain the other (in this case, if $U_{x_i} \subsetneq U_{x_j}$, clearly we must have $a_i = a_j$ and we can remove U_{x_i} from our subcover of X). We can therefore assume that all U_{x_i} are disjoint, and that $f = a_i$ on each U_{x_i}, and so the conclusion follows easily. "$\Leftarrow$" If $x \in \bigcup_{i=1}^n U_i$, then $x \in U_i$ for some i, and since U_i is open, there is a ball $x \in V \subsetneq U_i$ on which $f = ka_i$, where k is the number of sets U_i that contain x. If $x \notin \bigcup_{i=1}^n U_i$, then $f(x) = 0$ and, since a finite union of compact sets is compact, hence closed (in the metric space $\mathbb{Q}_p$), its complement is open and one finds a ball V around x on which $f = 0$, proving that f is locally constant. $\square$

Problem 18 (10 pts.)

Let k be a field and let $|\ |$ be a non-Archimedean valuation on k. Prove that the set

$$\mathcal{O} = \overline{U}_1(0) = \{x \in k : |x| \leq 1\}$$

is a subring of k. Prove further that its subset

$$\mathfrak{P} = U_1(0) = \{x \in k : |x| < 1\}$$

is an ideal of $\mathcal{O}$. Finally, show that $\mathfrak{P}$ is a maximal ideal in $\mathcal{O}$ and every element of the complement $\mathcal{O} - \mathfrak{P}$ is invertible in $\mathcal{O}$.

Solution. To check that $\mathcal{O}$ is a subring of k, we only need to check that $0, 1 \in \mathcal{O}$ and that $\mathcal{O}$ is closed under addition (which follows by the strong triangle inequality), multiplication (clear) and change of sign (also clear). To show that $\mathfrak{P}$ is an ideal of $\mathcal{O}$, we need to check that $0 \in \mathfrak{P}$, that $\mathfrak{P}$ is closed under addition (which again follows by the strong triangle inequality) and that if $x \in \mathcal{O}$ and $y \in \mathfrak{P}$, then $xy \in \mathfrak{P}$ (easy, since $|x| \leq 1, |y| < 1 \Longrightarrow |xy| < 1$). Next, if $x \in \mathcal{O} \setminus \mathfrak{P}$, then $|x| = 1$, hence $|1/x| = 1$ and $1/x \in \mathcal{O}$, i.e., x is invertible in $\mathcal{O}$, which means $\mathfrak{P}$ is a maximal ideal of $\mathcal{O}$ and that every element in $\mathcal{O} \setminus \mathfrak{P}$ is invertible. $\square$

The following exercises will be discussed in the tutorial and you do not need to hand in solutions for them.

Exercise 18

Give an example of a noncompact open subset of $\mathbb{Z}_p$.

Solution. The complement of any point $\{x\}$. For if it were compact, then it would also be closed, hence $\{x\}$ would be open, which is false. $\square$

Exercise 19

Let U be an open subset of a topological space X. Show that the characteristic function $f : X \to \mathbb{Z}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

is locally constant if $X = \mathbb{Z}_p$ and U is compact, but is not locally constant for any (open) set U if $X = \mathbb{R}$ (unless $U = \mathbb{R}$ or $\varnothing$).

Solution. Let $X = \mathbb{Z}_p$. Since U is open, if $x \in U$, there exists a ball $V \subset U$ around x and $f(x) = 1$ on V. Next, since U is a compact subset of the metric space $\mathbb{Z}_p$, it is also closed, hence $\mathbb{Z}_p \setminus U$ is open and we can apply the same reasoning. If $X = \mathbb{R}$ and, by contradiction, there is such a locally constant function f, then $\mathbb{R} \setminus U$ must be clopen, but the only clopen subsets of $\mathbb{R}$ are $\varnothing$ and $\mathbb{R}$ itself. Note that $\mathbb{Z}_p$ itself is compact, while $\mathbb{R}$ is not. □

Exercise 20

Prove Proposition 1.19 from the Script and compare it with Proposition 1.20. Why doesn't the proof work as easily for Proposition 1.20?

Solution. This is merely a manipulation of formal power series, there are no convergence issues involved:

$$\begin{aligned}\exp_p(x+y) &= \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{n!} \frac{n!}{(n-k)!k!} x^{n-k} y^k \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^{n-k}}{(n-k)!} \frac{y^k}{k!} = \left(\sum_{m=0}^{\infty} \frac{x^m}{m!} \right) \left(\sum_{k=0}^{\infty} \frac{y^k}{k!} \right) = \exp_p(x) \exp_p(y).\end{aligned}$$ □