Homework 6

The homeworks are due on the Thursday of the week after the assignment was posted online[1]. Please hand in your homework at the beginning of the tutorial or bring it to the lecture on Thursday morning. You can work on and submit your homework in groups of two. Please staple your pages and write your names and matriculation numbers on the first page.

Problem 16 (10 pts.)

(i) Use power series to define p-adic analogues "$\sin_p$" and "$\cos_p$" of the usual sin and cos functions, and determine their regions of convergence.

(ii) Show that if $p \equiv 1 \pmod 4$, then there exists $i \in \mathbb{Q}_p$ such that $i^2 = -1$ and the classical relation

$$\exp_p(ix) = \cos_p(x) + i \sin_p(x)$$

holds for any x in the convergence region.

(iii) We know that the classical trigonometric functions sin and cos are periodic. Are the p-adic analogues also periodic?

Problem 17 (10 pts.)

Let $X \subset \mathbb{Q}_p$ be a compact and open subset. Show that a function $f : X \to \mathbb{Q}_p$ is locally constant if and only if there are finitely many compact and open subsets $U_1, \ldots, U_n \subset X$ such that

$$f = \sum_{i=1}^{n} a_i \mathbf{1}_{U_i}$$

with $a_i \in \mathbb{Q}_p$. Here, $\mathbf{1}_{U_i} : X \to \{0, 1\}$ is the characteristic function of U_i, i.e $\mathbf{1}_{U_i}(x) = 1$ if $x \in U_i$ and $\mathbf{1}_{U_i}(x) = 0$ otherwise.

Problem 18 (10 pts.)

Let k be a field and let $| \ |$ be a non-Archimedean valuation on k. Prove that the set

$$\mathcal{O} = \overline{U}_1(0) = \{x \in k : |x| \leq 1\}$$

is a subring of k. Prove further that its subset

$$\mathfrak{P} = U_1(0) = \{x \in k : |x| < 1\}$$

is an ideal of $\mathcal{O}$. Finally, show that $\mathfrak{P}$ is a maximal ideal in $\mathcal{O}$ and every element of the complement $\mathcal{O} - \mathfrak{P}$ is invertible in $\mathcal{O}$.

[1] This assignment is due Thursday, 21.11.19.

The following exercises will be discussed in the tutorial and you do not need to hand in solutions for them.

Exercise 18

Give an example of a noncompact open subset of $\mathbb{Z}_p$.

Exercise 19

Let U be an open subset of a topological space X. Show that the characteristic function $f : X \to \mathbb{Z}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

is locally constant if $X = \mathbb{Z}_p$ and U is compact, but is not locally constant for any (open) set U if $X = \mathbb{R}$ (unless $U = \mathbb{R}$ or $\varnothing$).

Exercise 20

Prove Proposition 1.19 from the Script and compare it with Proposition 1.20. Why doesn't the proof work as easily for Proposition 1.20?