Homework 5

The homeworks are due on the Thursday of the week after the assignment was posted online[1]. Please hand in your homework at the beginning of the tutorial or bring it to the lecture on Thursday morning. You can work on and submit your homework in groups of two. Please staple your pages and write your names and matriculation numbers on the first page.

Problem 13 (10 pts.)

Prove Proposition 1.15 from the Script.

Solution. If $x < \rho$, then $\sum |a_n||x^n|$ converges as a power series in $\mathbb{R}$, therefore it converges as a p-adic series too. If $|x| > \rho$, it is easy to see that $|a_n||x^n|$ cannot tend to 0 as $n \to \infty$ since, by definition, $|a_n|$ is close to $1/\rho^n$ and therefore $(|x|/\rho)^n$ gets arbitrarily large as $n \to \infty$. That $f(0)$ converges is trivial. Thus, all that is left to deal with is the case when $|x| = \rho$ in parts (iii) and (iv), and this follows from Corollary 1.3 from the Script. $\square$

Problem 14 (10 pts.)

Complete the proof of Proposition 1.13 from the Script, by showing that if a, b, c are pairwise coprime square-free integers, the equation

$$aX^2 + bY^2 + cZ^2 = 0$$

has non-trivial solutions in $\mathbb{Q}_2$ if and only if the following conditions are satisfied:

(i) if a, b, c are all odd, then there are two of them whose sum is divisible by 4.

(ii) if a is even, then either $b + c$ or $a + b + c$ is divisible by 8 (similarly if one of b, c is even).

Solution. (i) "$\Leftarrow$" We claim that the equation $ax^2 + by^2 + cz^2 \equiv 0 \pmod 8$ has a solution (x_0, y_0, z_0). Say that $4 \mid a + b$. There are now two posibilities: 1. if $a + b \equiv 0 \pmod 8$, choose $x_0 = y_0 = 1$ and $z_0 = 0$; 2. if $a + b \equiv 4 \pmod 8$, choose $x_0 = y_0 = 1$ and $z_0 = 2$. Now that the claim is proven, let $f(X) = aX^2 + by_0^2 + cz_0^2$. The idea is to use the strong Hensel's lemma. We have $|f'(x_0)|_2^2 = |2ax_0|_2^2 = 2^{-2} > 2^{-3} \geq |f(x_0)|_2 = |ax_0^2 + by_0^2 + cz_0^2|_2$ and this gives a solution (x, y_0, z_0) in $\mathbb{Q}_2$. "$\Rightarrow$" Conversely, suppose the equation has a solution (x, y, z) in $\mathbb{Q}_2$. We can assume (eventually by multiplying by a power of 2) that $\max\{|x|_2, |y|_2, |z|_2\} = 1$, i.e., that $x, y, z \in \mathbb{Z}_2$ are not *all* in $2\mathbb{Z}_2$. Reducing modulo $2\mathbb{Z}_2$ we see that exactly two of x, y, z will be 2-adic units, say y and z, while the other one will be divisible by 2. As the square of a 2-adic unit will be in $1 + 4\mathbb{Z}_2$ and that of an element in $2\mathbb{Z}_2$ will be in $4\mathbb{Z}_2$, reducing modulo $4\mathbb{Z}_2$ yields $b + c \equiv 0 \pmod 4$. (ii) "$\Leftarrow$" Similarly as in the previous part, an (even easier) application of the stronger Hensel's lemma will do.

[1] This assignment is due Thursday, 14.11.19.

"$\Rightarrow$" Suppose a is even, in which case b, c must be odd. As before, we can assume that at least one of x, y, z is a 2-adic unit, and that all three are in $\mathbb{Z}_2$. There are two cases to be considered: 1. if $x \in 2\mathbb{Z}_2$, then $2x^2 \equiv 0 \pmod 8$ and y, z must be 2-adic units, and so their squares are both in $1 + 8\mathbb{Z}_2$, from where it follows that $0 = ax^2 + by^2 + cz^2 \equiv b + c \pmod 8$; 2. if $x \in \mathbb{Z}_2^\times$, then also $y, z \in \mathbb{Z}_2^\times$ (for otherwise, if, say, $y \in 2\mathbb{Z}_2$, then $ax^2 + by^2 \equiv 0 \pmod 2$, and therefore $cz^2 \equiv 0 \pmod 2$ too, hence, since c is odd, $z \in 2\mathbb{Z}_2$; but then $by^2 + cz^2 \in 4\mathbb{Z}_2$, hence also $ax^2 \in 4\mathbb{Z}_2$, a contradiction). Again, as a square of a 2-adic unit is always 1 modulo 8, we get $a + b + c \equiv 0 \pmod 8$, which proves the claim in this case as well. $\square$

Problem 15 (10 pts.)

Find the exact domain of convergence (with respect to $|\ |_p$ for a prime p) of the following series.

(i) $\sum_{n=0}^\infty n! X^n$ (ii) $\sum_{n=0}^\infty p^n X^n$ (iii) $\sum_{n=0}^\infty p^n X^{p^n}$

Solution. (i) We need to compute $\nu_p(n!x^n) = \nu_p(n!) + n\nu_p(x)$. To estimate $\nu_p(n!)$, we recall Exercise 12, which says that

$$\nu_p(n!) = \frac{n - S}{p - 1} < \frac{n}{p - 1},$$

where S is the sum of the p-adic digits of n. If $\nu_p(x) > -\frac{1}{p-1}$, say $\nu_p(x) = -\frac{1}{p-1} + \alpha$, with $\alpha > 0$, then $\nu_p(n!x^n) = n\alpha - \frac{S}{p-1} \geq n\alpha - \lfloor \log_p n \rfloor - 1$, hence $\lim\limits_{n\to\infty} \nu_p(n!x^n) = \infty$ and the series converges in the open ball of radius $p^{1/(p-1)}$. It can be further seen that $\rho \leq p^{1/(p-1)}$ and the series does not converge on the boundary, since, for $|x|_p = p^{1/(p-1)}$ and $n = p^m$ we see that $\nu_p(n!x^n) = \frac{p^m-1}{p-1} - \frac{p^m}{p-1} = -\frac{1}{p-1}$ does not converge to 0 as $n \to \infty$. (ii) We have $\nu_p(p^n x^n) = n + n\nu_p(x)$. If $\nu_p(x) > -1$, this tends to ∞ so that the series will converge; otherwise, the series will diverge. Thus, the domain of convergence is the open ball of radius p. (iii) Similarly, we have $\nu_p(p^n x^{p^n}) = n + p^n \nu_p(x)$. If $\nu_p(x) \geq 0$, then this will tend to ∞ and the series will converge; otherwise, the series will diverge. Thus, the domain of convergence is the closed ball of radius 1. $\square$

The following exercises will be discussed in the tutorial and you do not need to hand in solutions for them.

Exercise 16

Decide if the following sequences converge in $\mathbb{Q}_p$ and find the limit of those that do:

(i) $a_n = n!$ (ii) $a_n = n$ (iii) $a_n = 1/n$ (iv) $a_n = p^n$ (v) $a_n = (1+p)^{p^n}$

Exercise 17

In this problem $\log_p$ denotes the p-adic logarithm defined in class.

1. Prove that $\log_2(-1)$ exists and is equal to 0.
2. Show that $\log_2 x = 0$ if and only if $x = \pm 1$.
3. Further, use this to show that there are no fourth roots of unity in $\mathbb{Q}_2$.
4. Show that for $p \neq 2$, we have $\log_p(x) = 0$ if and only if $x = 1$.