Homework 5

The homeworks are due on the Thursday of the week after the assignment was posted online[1]. Please hand in your homework at the beginning of the tutorial or bring it to the lecture on Thursday morning. You can work on and submit your homework in groups of two. Please staple your pages and write your names and matriculation numbers on the first page.

Problem 13 (10 pts.)

Prove Proposition 1.15 from the Script.

Problem 14 (10 pts.)

Complete the proof of Proposition 1.13 from the Script, by showing that if a, b, c are pairwise coprime square-free integers, the equation

$$aX^2 + bY^2 + cZ^2 = 0$$

has non-trivial solutions in $\mathbb{Q}_2$ if and only if the following conditions are satisfied:

(i) if a, b, c are all odd, then there are two of them whose sum is divisible by 4.

(ii) if a is even, then either $b + c$ or $a + b + c$ is divisible by 8 (similarly if one of b, c is even).

Problem 15 (10 pts.)

Find the exact domain of convergence (with respect to $| \, |_p$ for a prime p) of the following series.

(i) $\sum_{n=0}^{\infty} n! X^n$ (ii) $\sum_{n=0}^{\infty} p^n X^n$ (iii) $\sum_{n=0}^{\infty} p^n X^{p^n}$

The following exercises will be discussed in the tutorial and you do not need to hand in solutions for them.

Exercise 16

Decide if the following sequencens converge in $\mathbb{Q}_p$ and find the limit of those that do:

(i) $a_n = n!$ (ii) $a_n = n$ (iii) $a_n = 1/n$ (iv) $a_n = p^n$ (v) $a_n = (1+p)^{p^n}$

[1] This assignment is due Thursday, 14.11.19.

Exercise 17

Let $f(X) = \sum_{n=0}^{\infty} a_n X^n$ be a (formal) power series with coefficients in $a_n \in \mathbb{Z}_p$. Then f converges on $U_1(0)$.

Exercise 18

In this problem $\log_p$ denotes the p-adic logarithm defined in class.

1. Prove that $\log_2(-1)$ exists and is equal to 0.
2. Show that $\log_2 x = 0$ if and only if $x = \pm 1$.
3. Further, use this to show that there are no fourth roots of unity in $\mathbb{Q}_2$.
4. Show that for $p \neq 2$, we have $\log_p(x) = 0$ if and only if $x = 1$.