Homework 3

The homeworks are due on the Thursday of the week after the assignment was posted online[1]. Please hand your homework in at the beginning of the tutorial at 16:00 or bring it to the lecture on Thursday morning. You can work on and submit your homework in groups of two. Please staple your pages and write your names and matriculation numbers on the first page.

Problem 7 (10 pts.)

Find the p-adic expansion of

(i) $(6 + 4 \cdot 7 + 2 \cdot 7^2 + 1 \cdot 7^3 + \cdots)(3 + 0 \cdot 7 + 0 \cdot 7^2 + 6 \cdot 7^3 + \cdots)$ in $\mathbb{Q}_7$ to 4 digits;

(ii) $1/(3 + 2 \cdot 5 + 3 \cdot 5^2 + 1 \cdot 5^3 + \cdots)$ in $\mathbb{Q}_5$ to 4 digits;

(iii) $\frac{2}{3}$ in $\mathbb{Q}_2$;

(iv) $-\frac{1}{6}$ in $\mathbb{Q}_7$;

(v) $\frac{1}{3!}$ in $\mathbb{Q}_3$.

Solution. (i) $4+1\cdot 7^2+5\cdot 7^3$ (in digit notation: $4,015$) (ii) $2+1\cdot 5^2+3\cdot 5^3$ (in digit notation: $2,013$) (iii) $1 \cdot 2 + \sum_{n=1}^{\infty} 2^{2n}$ (in digit notation: $0,1\overline{10}$) (iv) $-\frac{1}{6} = \frac{1}{1-7} = \sum_{n=0}^{\infty} 7^n$ (in digit notation: $1,\overline{1}$) (v) $\frac{1}{3!} = -3^{-1} \cdot \frac{1}{1-3} = -3^{-1} \cdot (1+3+3^2+\cdots) = 2 \cdot 3^{-1} + 1 + 1 \cdot 3 + 1 \cdot 3^2 + \cdots$ (in digit notation: $21,\overline{1}$) $\square$

Problem 8 (10 pts.)

A *neighborhood* of a point x in a topological space is a set that contains an open ball containing x. A *fundamental system of neighborhoods* of a point x is a family $(U_i)_{i\in I}$ of neighborhoods of x with the property that any neighborhood of x contains one of the U_i.

Prove that the sets $p^n\mathbb{Z}_p$, $n \in \mathbb{Z}$, form a fundamental system of neighborhoods of 0 in $\mathbb{Q}_p$ and that we have

$$\mathbb{Q}_p = \bigcup_{n\in\mathbb{Z}} p^n\mathbb{Z}_p.$$

Solution. Since $\mathbb{Z}_p$ is the closed unit ball around 0, it is also an open set containing 0 (hence a neighborhood of 0). It is easy to check that the map $\mathbb{Q}_p \to \mathbb{Q}_p$ given by $x \mapsto px$ is a homeomorphism (that is, a continuous map with a continuous inverse). As such, multiplication by p sends open sets to open sets, which means that, for every $n \in \mathbb{Z}$, the set $p^n\mathbb{Z}_p$ is a neighborhood of 0. The fact that

$$\mathbb{Q}_p = \bigcup_{n\in\mathbb{Z}} p^n\mathbb{Z}_p$$

[1] This assignment is due Thursday, 31.10.19.

is clear from what we know by now about $\mathbb{Z}_p$ and $\mathbb{Q}_p$. To show that these sets form a fundamental system of neighborhoods, we need to prove that any open set containing 0 contains some $p^n\mathbb{Z}_p$ which is clear since any open ball containing 0 contains a closed ball of smaller radius, and this will be one of the $p^n\mathbb{Z}_p$. □

Problem 9 (10 pts.)

(i) You know from class that $\mathbb{Q}_p$ is a totally disconnected topological space. Prove that $\mathbb{Q}_p$ is a Hausdorff space.

(ii) Prove that $\mathbb{Z}_p$ is compact and that $\mathbb{Q}_p$ is locally compact. (A topological space is called *locally compact* if every point has a neighborhood which is a compact set.)
Hint: You might want to use (and prove) the following fact: If k is a field with a norm, show that k is locally compact in the topology induced by that norm if and only if there exists a neighborhod of 0 that is compact (if a set X is compact, the translated sets $\{a + x : x \in X\}$ are images of X under a continuous map, hence also compact).

Solution. (i) For any $a \neq b \in \mathbb{Q}_p$, take n large enough so that $p^{-n} < |a - b|_p$. The balls of radius p^{-n} around a and b are disjoint (easy to see with the strong triangle inequality).
(ii) Using the hint, we see that, since $\mathbb{Z}_p$ is a neighborhood of 0 in $\mathbb{Q}_p$, proving that $\mathbb{Z}_p$ is compact is the same with proving that $\mathbb{Q}_p$ is locally compact. Therefore, it is enough to prove the former. As a closed set in the complete field $\mathbb{Q}_p$, we know that $\mathbb{Z}_p$ is also complete. Recalling some basic topology, to show that $\mathbb{Z}_p$ is compact, it is enough then to prove that it is totally bounded, i.e., that we can cover $\mathbb{Z}_p$ with finitely many balls for radius ε, for any $\varepsilon > 0$. Certainly it is enough to check this for $\varepsilon = p^{-n},\ n \geq 0$. One can prove without much difficulty that $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p^n\mathbb{Z}$ and that the cosets of $p^n\mathbb{Z}_p$ in $\mathbb{Z}_p$ are also balls in the p-adic topology. This means that, as a ranges through $0, 1, \ldots, p^n - 1$, the p^n balls

$$a + p^n\mathbb{Z}_p = \{a + p^n x : x \in \mathbb{Z}_p\} = \{y \in \mathbb{Z}_p : |y - a| \leq p^n\} = \overline{U}_a(p^{-n})$$

cover $\mathbb{Z}_p$ (which can certainly be seen also without the isomorphism above). □

The following exercises will be discussed in the tutorial and you do not need to hand in solutions for them.

Exercise 10

A p-adic integer $a = a_0 + a_1p + a_2p^2 + \cdots$ is a unit in the ring $\mathbb{Z}_p$ if and only if $a_0 \neq 0$.

Solution. If there is $b = \sum_{n\geq 0} b_n p^n$ such that $ab = 1$, then we must have $a_0b_0 \equiv 1 \pmod p$, which forces $a_0 \neq 0$. Conversely, if $a_0 \neq 0$, we can either directly construct an inverse $a^{-1} \in \mathbb{Z}_p$ by choosing $0 < b_0 \leq p - 1$ with $a_0b_0 \equiv 1 \pmod p$ and $a^{-1} = b_0 + b_1p + \cdots$, where $0 \leq b_1 \leq p - 1$ is chosen such that $a_1b_0 + t + a_0b_1 \equiv 0 \pmod p$, with $a_0b_0 = 1 + tp$, etc. or note that if $a_0 \neq 0$, then $|a|_p = 1$, hence $a \in \mathbb{Z}_p^\times$ by Lemma 4.1 from the Script. □

Exercise 11

If $a \in \mathbb{Q}_p$ has the p-adic expansion $a_{-m}p^{-m} + a_{-m+1}p^{-m+1} + \cdots + a_0 + a_1p + \cdots$, what is the p-adic expansion of $-a$?

Solution. $-a = (p - a_{-m})p^{-m} + (p - 1 - a_{-m+1})p^{-m+1} + \cdots + (p - 1 - a_0) + (p - 1 - a_1)p + \cdots$. $\square$

Exercise 12

Prove that if $n = a_0 + a_1 p + a_2 p^2 + \cdots + a_s p^s$ is written in base p, so that $0 \leq a_i \leq p - 1$, and if $S = \sum_{i=1}^{s} a_0$ (S is the sum of the p-adic digits of $n \in \mathbb{N}$), then we have

$$\nu_p(n!) = \frac{n - S}{p - 1}.$$

Solution. We know from Legendre's theorem (but this is just an easy counting) that

$$\nu_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor.$$

We thus have

$$\begin{aligned} \nu_p(n!) &= \sum_{i=1}^{s} \left\lfloor \frac{a_0 + \cdots + a_{i-1}p^{i-1} + a_i p^i + \cdots + a_s p^s}{p^i} \right\rfloor = \sum_{i=1}^{s} a_i (p^{i-1} + \cdots + p + 1) \\ &= \sum_{i=1}^{s} a_i \frac{p^i - 1}{p - 1} = \frac{\sum_{i=1}^{s} a_i p^i - \sum_{i=1}^{s} a_i}{p - 1} = \frac{\sum_{i=0}^{s} a_i p^i - \sum_{i=0}^{s} a_i}{p - 1} = \frac{n - S}{p - 1}. \end{aligned}$$ $\square$