Homework 3

The homeworks are due on the Thursday of the week after the assignment was posted online[1]. Please hand in your homework at the beginning of the tutorial at 16:00 or bring it to the lecture on Thursday morning. You can work on and submit your homework in groups of two. Please staple your pages and write your names and matriculation numbers on the first page.

Problem 7 (10 pts.)

Find the p-adic expansion of

(i) $(6 + 4 \cdot 7 + 2 \cdot 7^2 + 1 \cdot 7^3 + \cdots)(3 + 0 \cdot 7 + 0 \cdot 7^2 + 6 \cdot 7^3 + \cdots)$ in $\mathbb{Q}_7$ to 4 digits;

(ii) $1/(3 + 2 \cdot 5 + 3 \cdot 5^2 + 1 \cdot 5^3 + \cdots)$ in $\mathbb{Q}_5$ to 4 digits;

(iii) $\frac{2}{3}$ in $\mathbb{Q}_2$;

(iv) $-\frac{1}{6}$ in $\mathbb{Q}_7$;

(v) $\frac{1}{3!}$ in $\mathbb{Q}_3$.

Problem 8 (10 pts.)

A *neighborhood* of a point x in a topological space is a set that contains an open ball containing x. A *fundamental system of neighborhoods* of a point x is a family $(U_i)_{i\in I}$ of neighborhoods of x with the property that any neighborhood of x contains one of the U_i. Prove that the sets $p^n\mathbb{Z}_p$, $n \in \mathbb{Z}$, form a fundamental system of neighborhoods of 0 in $\mathbb{Q}_p$ and that we have

$$\mathbb{Q}_p = \bigcup_{n\in\mathbb{Z}} p^n\mathbb{Z}_p.$$

Problem 9 (10 pts.)

(i) You know from class that $\mathbb{Q}_p$ is a totally disconnected topological space. Prove that $\mathbb{Q}_p$ is a Hausdorff space.

(ii) Prove that $\mathbb{Z}_p$ is compact and that $\mathbb{Q}_p$ is locally compact. (A topological space is called *locally compact* if every point has a neighborhood which is a compact set.)
Hint: You might want to use (and prove) the following fact: If k is a field with a norm, show that k is locally compact in the topology induced by that norm if and only if there exists a neighborhod of 0 that is compact (if a set X is compact, the translated sets $\{a + x : x \in X\}$ are images of X under a continuous map, hence also compact).

[1] This assignment is due Thursday, 31.10.19.

The following exercises will be discussed in the tutorial and you do not need to hand in solutions for them.

Exercise 10

A p-adic integer $a = a_0 + a_1 p + a_2 p^2 + \cdots$ is a unit in the ring $\mathbb{Z}_p$ if and only if $a_0 \neq 0$.

Exercise 11

If $a \in \mathbb{Q}_p$ has the p-adic expansion $a_{-m}p^{-m} + a_{-m+1}p^{-m+1} + \cdots + a_0 + a_1 p + \cdots$, what is the p-adic expansion of $-a$?

Exercise 12

Prove that if $n = a_0 + a_1 p + a_2 p^2 + \cdots + a_s p^s$ is written in base p, so that $0 \leq a_i \leq p-1$, and if $S = \sum_{i=0}^{s} a_i$ (S is the sum of the p-adic digits of $n \in \mathbb{N}$), then we have

$$\nu_p(n!) = \frac{n - S}{p - 1}.$$