Homework 2

The homeworks are due on the Thursday of the week after the assignment was posted online[1]. Please hand your homework in at the beginning of the tutorial or bring it to the lecture on Thursday morning. You can work on and submit your homework in groups of two. Please staple your pages and write your names and matriculation numbers on the first page.

Problem 4 (10 pts.)

Find, for any prime p, a Cauchy sequence of rational numbers that does not converge to a rational number with respect to $|\ |_p$.
Hints: For $p \neq 2$ choose an integer $a \in \mathbb{Z}$ such that

- a is not a square in $\mathbb{Q}$;
- $p \nmid a$;
- a is a quadratic residue modulo p, i.e., the congruence $x^2 \equiv a \pmod{p}$ has a solution.

First, convince yourself that such an a exists. Second, try to construct a Cauchy sequence $(x_n)_{n\geq 0}$ starting with terms x_0 and x_1 that satisfy $x_0^2 \equiv a \pmod{p}$, $x_1 \equiv x_0 \pmod{p}$ and $x_1^2 \equiv a \pmod{p^2}$. Iterate this process to obtain a Cauchy sequence that converges to a number that is not in $\mathbb{Q}$. Along this process, you can also use Exercise 6.

Bonus (5 pts.): Do the same for $p = 2$.

Solution. Such an a is easy to construct. For instance, take $a = 1 + p$. Pick x_0 such that $x_0^2 \equiv a \pmod{p}$. We want to find an x_1 satisfying $x_1 \equiv x_0 \pmod{p}$ and $x_1^2 \equiv a \pmod{p^2}$. Choosing $x_1 = x_0 + kp$ with $k \in \mathbb{Z}$, the first condition is satisfied. For the second condition, we need $x_1^2 = x_0^2 + 2x_0kp \equiv a \pmod{p^2}$. But $x_0^2 = a + tp$, for some $t \in \mathbb{Z}$. Therefore $x_1^2 \equiv a + p(t + 2x_0k) \pmod{p^2}$, and we need to find a k such that $t + 2x_0k \equiv 0 \pmod{p}$. But such a k certainly exists, as $(p, 2x_0) = 1$. This process can be iterated to yield a sequence (x_n) satisfying

$$x_n \equiv x_{n-1} \pmod{p^n} \quad \text{and} \quad x_n^2 \equiv a \pmod{p^{n+1}}.$$

Since

$$|x_{n+1} - x_n| = |\lambda p^{n+1}| \leq p^{-(n+1)} \to 0,$$

for some $\lambda \in \mathbb{Z}$, Exercise 6 tells us that this sequence is Cauchy. On the other hand, we have

$$|x_n^2 - a| = |\mu p^{n+1}| \leq p^{-(n+1)} \to 0,$$

for some $\mu \in \mathbb{Z}$, hence the limit, if it existed, would be a square root of a. However, a is not a square in $\mathbb{Q}$ (therefore $\mathbb{Q}$ is not complete with respect to $|\ |_p$).
For the bonus part, i.e., the case $p = 2$, one picks an a such that

[1] This assignment is due Thursday, 24.10.19.

- a is not a cube in $\mathbb{Q}$;
- $2 \nmid a$;
- a is a cubic residue modulo p, i.e., the congruence $x^3 \equiv a \pmod 2$ has a solution.

To simplify things, let us notice that such a number is given by $a = 3$. We construct, just like in the previous case, a sequence (x_n) such that

$$x_n \equiv x_{n-1} \pmod{2^n} \quad \text{and} \quad x_n^3 \equiv 3 \pmod{2^{n+1}}.$$

For this, it is enough to have the starting terms of the sequence. To find them, pick x_0 satisfying $x_0^3 = 3 \pmod 2$. We need an x_1 such that $x_1 \equiv x_0 \pmod 2$ and $x_1^2 \equiv 3 \pmod{2^2}$. The first condition is satisfied if we take $x_1 = x_0 + 2k$ with $k \in \mathbb{Z}$. For the second condition, we need $x_1^3 = x_0^3 + 6x_0^2k + 12x_0k^2 + 2^3k^3 \equiv x_0^3 + 6x_0^2k \equiv 3 \pmod{2^2}$. But we know that $x_0^3 = 3 + 2t$ for some $t \in \mathbb{Z}$, by assumption. Therefore, the congruence becomes equivalent with proving that $t + 3x_0^2k \equiv 0 \pmod 2$ has a solution for t, which is again clear. One now iterates this process to find a Cauchy sequence (x_n) that would converge to $\sqrt[3]{3} \notin \mathbb{Q}$. $\square$

Problem 5 (10 pts.)

(i) Give $\mathbb{Q}$ the 5-adic topology and consider the triangle whose vertices are $x = 2/15$, $y = 1/5$ and $z = 7/15$. What are the lengths of the three sides? Justify your answer.

(ii) Take the 5-adic absolute value on $\mathbb{Q}$. Show that $B(1,1) = B(1,1/2) = \overline{B}(1,1/5)$. Can you justify in a few words why?

Solution. (i) $|x-y|_5 = |-1/15|_5 = 5$, $|y-z|_5 = |-4/15|_5 = 5$, $|z-x|_5 = |1/3|_5 = 1$.

(ii) The 5-adic absolute value can only take values of the form 5^n with $n \in \mathbb{Z}$. So saying that a 5-adic absolute value is less than 1, less than 1/2 or less than or equal to 1/5 all amount to the same thing, and this is why $B(1,1) = B(1,1/2) = \overline{B}(1,1/5)$. $\square$

Problem 6 (10 pts.)

Show that $x^2 = 6$ has a solution in $\mathbb{Q}_5$.

Solution. The idea is similar to that from Problem 4. The equation $x^2 \equiv 6 \pmod 5$ has a solution, namely $x \equiv 1 \pmod 5$. We show inductively that $x^2 \equiv 6 \pmod{5^n}$ has a solution for every $n \geq 1$. If we have $x_n^2 \equiv 6 \pmod{5^n}$, we want to find $x_{n+1}^2 \equiv 6 \pmod{5^n}$. For this, write $x_{n+1} = x_n + 5^n t$, and we must have $(x_n + 5^n t)^2 \equiv 6 \pmod{5^{n+1}}$, which means $2 \cdot 5^n t x_n + x_n^2 \equiv 6 \pmod{5^{n+1}}$. This reduces to

$$2tx_n + \frac{x_n^2 - 6}{5^n} \equiv 0 \pmod 5,$$

which we can solve for t, as $(5, 2x_n) = 1$. This produces a sequence (x_n) such that $x_n^2 \equiv 6 \pmod{5^n}$ and $x_{n+1} \equiv x_n \pmod{5^n}$. As in Problem 4, we see that this sequence is Cauchy and that it converges to a limit x which is (by completeness) in $\mathbb{Q}_p$ and which will be the solution to our equation. $\square$

The following exercises will be discussed in the tutorial and you do not need to hand in solutions for them.

Exercise 5

You have defined in class $|\ |_p$ on $\mathbb{Q}_p$. Prove that this is a non-Archimedean norm on $\mathbb{Q}_p$.

Solution. The fact that $x \in \mathbb{Q}_p$ satisfies $|x|_p = 0$ if and only if $x = 0$ is clear from the definition. Take any $x, y \in \mathbb{Q}_p$, represented by the Cauchy sequences $(x_n), (y_n)$. Then $|xy|_p = \lim_{n\to\infty} |x_n y_n|_p = \lim_{n\to\infty} |x_n|_p |y_n|_p = \lim_{n\to\infty} |x_n| \lim_{n\to\infty} |y_n|_p = |x|_p |y|_p$. We are only left to check the strong triangle inequality. If any of the two Cauchy sequences representing x and y tends to 0, then everything is trivial, so assume that is not the case. But then the two sequences will become stationary, so there is N such that $|x_n|_p = |x|_p$ and $|y_n|_p = |y|_p$ for any $n \geq N$, therefore the maximum between the two will also become stationary. If this is given by, say $|y|_p$, then we have

$$|x + y|_p = \lim_{n\to\infty} |x_n + y_n|_p \leq \lim_{n\to\infty} \max\{|x_n|_p, |y_n|_p\} = |y|_p = \max\{|x|_p, |y|_p\}.$$

Exercise 6

A sequence $(x_n)_{n\geq 0}$ of rational numbers is a Cauchy sequence with respect to a non-Archimedean absolute value $|\ |$ if and only if we have

$$\lim_{n\to\infty} |x_{n+1} - x_n| = 0.$$

Solution. One implication is trivial. For the other, for any $m > n$, write $m = n + r$ and

$$\begin{aligned}|x_m - x_n| &= |(x_{n+r} - x_{n+r-1}) + (x_{n+r-1} - x_{n+r-2}) + \cdots + (x_{n+1} - x_n)| \\ &\leq \max\{|(x_{n+r} - x_{n+r-1}|, |x_{n+r-1} - x_{n+r-2}|, \ldots, |x_{n+1} - x_n|\}\end{aligned}$$

as the absolute value $|\ |$ is non-Archimedean. The result is clear. □

Exercise 7

Show that $x^2 = 7$ has no solution in $\mathbb{Q}_5$.

Solution. If there were a solution, we could write x as a 5-adic number in the form

$$x = \sum_{n=-N}^{\infty} a_n 5^n.$$

The 5-adic expansion of 7 is $2 + 1 \cdot 5$, so that $N = 0$. Thus

$$x^2 = \left(\sum_{n=0}^{\infty} a_n 5^n\right)^2 = 2 + 1 \cdot 5,$$

from where we obtain $a_0^2 \equiv 2 \pmod 5$, but this equation has no solution. □

Exercise 8

Prove that if $x \in \mathbb{Q}$ and $|x|_p \leq 1$ for every prime p, then $x \in \mathbb{Z}$.

Solution. Write $x = \frac{a}{b}$. The fact that $|x|_p \leq 1$ means that $\nu_p(a) - \nu_p(b) \geq 0$, i.e., $\nu_p(a) \geq \nu_p(b)$ for any prime p. But this further means that $b \mid a$, thus $x \in \mathbb{Z}$. $\square$

Exercise 9

Find two non-zero Cauchy sequences with respect to the p-adic absolute value whose product is the zero sequence.
This shows that there are zero divisors in the ring of Cauchy sequences.

Solution. For example, if one takes

$$(s_n) = 0, p, 0, p^2, 0, p^3, 0, p^4, \ldots$$

and the shifted sequence

$$(t_n) = p, 0, p^2, 0, p^3, 0, p^4, \ldots,$$

then both (s_n) and (t_n) are Cauchy, and $s_n t_n = 0$. $\square$