Homework 2

The homeworks are due on the Thursday of the week after the assignment was posted online[1]. Please hand your homework in at the beginning of the tutorial at 16:00 or bring it to the lecture on Thursday morning. You can work on and submit your homework in groups of two. Please staple your pages and write your names and matriculation numbers on the first page.

Problem 4 (10 pts.)

Find, for any odd prime p, a Cauchy sequence of rational numbers that does not converge to a rational number with respect to $| \ |_p$.
Hints: Choose an integer $a \in \mathbb{Z}$ such that

- a is not a square in $\mathbb{Q}$;
- $p \nmid a$;
- a is a quadratic residue modulo p, i.e., the congruence $x^2 \equiv a \pmod{p}$ has a solution.

First, convince yourself that such an a exists. Second, try to construct a Cauchy sequence $(x_n)_{n\geq 0}$ starting with terms x_0 and x_1 that satisfy $x_0^2 \equiv a \pmod{p}$, $x_1 \equiv x_0 \pmod{p}$ and $x_1^2 \equiv a \pmod{p^2}$. Iterate this process to obtain a Cauchy sequence that converges to a number that is not in $\mathbb{Q}$. Along this process, you may also use Exercise 6.

Bonus (5 pts.): Do the same for $p = 2$.

Problem 5 (10 pts.)

(i) Give $\mathbb{Q}$ the 5-adic topology and consider the triangle whose vertices are $x = 2/15$, $y = 1/5$ and $z = 7/15$. What are the lengths of the three sides? Justify your answer.

(ii) Take the 5-adic absolute value on $\mathbb{Q}$. Show that $U_1(1) = U_{1/2}(1) = \overline{U}_{1/5}(1)$. Can you justify in a few words why?

Problem 6 (10 pts.)

Show that $x^2 = 6$ has a solution in $\mathbb{Q}_5$.

The following exercises will be discussed in the tutorial and you do not need to hand in solutions for them.

Exercise 5

You have defined in class $| \ |_p$ on $\mathbb{Q}_p$. Prove that this is a non-Archimedean norm on $\mathbb{Q}_p$.

[1] This assignment is due Thursday, 24.10.19.

Exercise 6

A sequence $(x_n)_{n\geq 0}$ of rational numbers is a Cauchy sequence with respect to a non-Archimedean absolute value $|\ |$ if and only if we have

$$\lim_{n\to\infty} |x_{n+1} - x_n| = 0.$$

Exercise 7

Show that $x^2 = 7$ has no solution in $\mathbb{Q}_5$.

Exercise 8

Prove that if $x \in \mathbb{Q}$ and $|x|_p \leq 1$ for every prime p, then $x \in \mathbb{Z}$.

Exercise 9

Find two non-zero Cauchy sequences with respect to the p-adic absolute value whose product is the zero sequence.
This shows that there are zero divisors in the ring of Cauchy sequences.