Homework 1

The homeworks are due on the Thursday of the week after the assignment was posted online[1]. You can either hand in your homework at the beginning of the tutorial or put it in the mailbox labeled "Algebraic Number Theory II" in Room 301 of the Mathematical Institute by 16:00. You can work on and submit your homework in groups of two. Please staple your pages and write your names and matriculation numbers on the first page.

Problem 1 (10 pts.)

Let $\| \ \|_1$ and $\| \ \|_2$ be two norms on a field F. Prove that $\| \ \|_1 \sim \| \ \|_2$ if and only if there exists $\alpha > 0$ such that $\|x\|_1 = \|x\|_2^\alpha$ for all $x \in F$.

Solution. If any of the two norms is trivial, then we have nothing to prove, so let us exclude the trivial norm from our discussion.
"$\Leftarrow$" If there is $\alpha > 0$ such that $\|x\|_1 = \|x\|_2^\alpha$ for all $x \in F$, the claim is clear, because this says that an open ball in $\| \ \|_1$ stays open in $\| \ \|_2$ and the other way around. Alternatively but equivalently, a Cauchy sequence in $\| \ \|_1$ stays Cauchy in $\| \ \|_2$ and conversely.
"$\Rightarrow$" For the other way around, assume that $\| \ \|_1 \sim \| \ \|_2$. For an arbitrary valuation $\| \ \|$ on F, the inequality $\|x\| < 1$ is equivalent to the fact that $(x^n)_{n\in\mathbb{N}} \to 0$ in the topology defined by $\| \ \|$. Translated to our norms, this says that if $\| \ \|_1 \sim \| \ \|_2$, we have

$$\|x\|_1 < 1 \iff \|x\|_2 < 1. \tag{1}$$

As F is a field and $\| \ \|_1$ is not the trivial norm, there exist elements $y \in F$ satisfying $\|y\|_1 > 1$. Fix such an element $y \in F$. Pick any other $x \in F,\ x \neq 0$. Certainly there is a real number a such that

$$\|x\|_1 = \|y\|_1^a.$$

We claim that this holds also in $\| \ \|_2$.
For this, pick a sequence $\{m_i/n_i\}_{i\in\mathbb{N}}$ of rational numbers with $n_i > 0$ that converges to a from above. This implies $\|x\|_1 = \|y\|_1^a < \|y\|_1^{m_i/n_i}$, hence

$$\left\| \frac{x^{n_i}}{y^{m_i}} \right\|_1 < 1, \quad \text{which implies also} \quad \left\| \frac{x^{n_i}}{y^{m_i}} \right\|_2 < 1$$

because of (1). Thus, we have $\|x\|_2 \leq \|y\|_2^{m_i/n_i}$, hence, taking limits, $\|x\|_2 \leq \|y\|_2^a$. Now, by taking a sequence $\{m_i/n_i\}_{i\in\mathbb{N}}$ of rational numbers with $n_i > 0$ that converges to a from below and doing the exact same argument, we obtain also $\|x\|_2 \geq \|y\|_2^a$. Therefore, we must have $\|x\|_2 = \|y\|_2^a$, and this holds for any $x \in F,\ x \neq 0$. Taking logarithms, we get

$$\frac{\log \|x\|_1}{\log \|x\|_2} = \frac{\log \|y\|_1}{\log \|y\|_2} := \alpha,$$

hence $\|x\|_1 = \|x\|_2^\alpha$ and in order for this to be really the α that we needed, we only have to check that $\alpha > 0$. This is clear since we assumed in the beginning that $\|y\|_1 > 1$, which also implies $\|y\|_2 > 1$ and therefore the logarithms above are all positive. $\square$

[1] This assignment is due Thursday, 17.10.19.

Problem 2 (10 pts.)

If $0 < \rho < 1$, prove that the function $\| \ \|$ defined on $\mathbb{Q}$ by

$$\|x\| = \begin{cases} \rho^{\nu_p(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is a non-Archimedean norm. Prove that this norm is equivalent to $| \ |_p$. What happens if $\rho = 1$? What about $\rho > 1$?

Solution. As $0 < \rho < 1$, there exists $\alpha > 0$ such that $\rho^{-1} = p^\alpha$, hence $\|x\| = (p^{-\nu_p(x)})^\alpha = |x|_p^\alpha$ and from here it follows at once that $\| \ \|$ is a non-Archimedean norm equivalent to $| \ |_p$. If $\rho = 1$ we just get the trivial norm. If $\rho > 1$, we do not get a norm any longer. For in this case there is N large enough so that, say, $\rho^N > 2$. If we take $x = 1$ and $y = p^N - 1$, then $\|x + y\| = \rho^{\nu_p(x+y)} = \rho^{\nu_p(p^N)} = \rho^N > 2 = 1 + 1 = \rho^0 + \rho^0 = \rho^{\nu_p(x)} + \rho^{\nu_p(y)} = \|x\| + \|y\|$, so the triangle inequality is not satisfied. $\square$

Problem 3 (10 pts.)

Product formula. Prove that for $x \in \mathbb{Q}$, $x \neq 0$,

$$\prod_p |x|_p = 1,$$

where the product is taken over all primes p, including $p = \infty$.

Solution. We only need to prove the formula for $x \in \mathbb{N}$, as the general case follows easily. Since any $x \in \mathbb{N}$ has a unique prime factor decomposition of the form $x = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ and, by definition,

$$\begin{cases} |x|_q = 1 & \text{if } q \neq p_i, \\ |x|_{p_i} = p_i^{-a_i} & \text{for } i = 1, 2, \ldots, k, \\ |x|_\infty = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} & \end{cases}$$

the result follows easily. $\square$

The following exercises will be discussed in the tutorial and you do not need to hand in solutions for them.

Exercise 1

Prove that $| \ |_p$ is not equivalent to $| \ |_q$ if p and q are distinct primes.

Solution. Assume the contrary. This means that there exists $\alpha > 0$ such that $|x|_p = |x|_q^\alpha$ for any $x \in \mathbb{Q}$. But for $x = p$ we obtain $|p|_p = p^{-1}$, while $|p|_q = 1$, hence we obtain a contradiction. The same holds if one of the primes is ∞. $\square$

Exercise 2

For $x \in \mathbb{Q}$ define $\|x\| = |x|^\alpha$ for a fixed positive real number α, where $| \ |$ denotes the usual absolute value. Show that $\| \ \|$ is a norm if and only if $\alpha \leq 1$ (in which case it is equivalent to $| \ |$.)

Solution. "$\Rightarrow$" If $\| \ \|$ is a norm, then $n^a|x|^a = |nx|^a \leq n|x|^a$, which implies $a \leq 1$.
"$\Leftarrow$" The other way around, assume $a \geq 1$. The only thing we need to check is

$$|x|^a + |y|^a \geq |x+y|^a.$$

WLOG we can assume $|x| \geq |y|$. Setting $u = y/x$, we have to show that

$$1 + |u|^a \geq |1+u|^a \quad \text{for } -1 \leq u \leq 1.$$

It is enough to prove that

$$1 + u^a \geq (1+u)^a \quad \text{for } 0 \leq u \leq 1.$$

If we let $f(u) = 1 + u^a - (1+u)^a$, we see that $f(0) = 0$, $f(1) \geq 0$ and $f'(u) > 0$ on $(0,1)$, which proves the claim. □

Exercise 3

Prove that two equivalent norms on a field F are either both Archimedean or both non-Archimedean.

Solution. Assume, by contradiction, that $\| \ \|_1$ is an Archimedean norm and that $\| \ \|_2$ is a non-Archimedean norm such that $\| \ \|_1 \sim \| \ \|_2$, which means that $\|x\|_1 = \|x\|_2^a$ for some fixed $a > 0$ and all $x \in F$. But then we know (see Prop. 1.3 from the Script) that $\|n\|_2 \leq 1$, hence $\|n\|_1 = \|n\|_2^a$ is bounded, contradiction with the fact that $\| \ \|_1$ is Archimedean. □

Exercise 4

Compute:

(i) $\nu_2(128)$

(ii) $\nu_2(128/7)$

(iii) $\nu_3(54)$

(iv) $\nu_3(10^9)$

(v) $\nu_3(7/9)$

(vi) $\nu_3(13.23)$

(vii) $\nu_7(-13.23)$

Solution. (i) 7 (ii) 7 (iii) 3 (iv) 0 (v) -2 (vi) 3 (vii) 2 □