Homework 13

The homeworks are due on the Thursday of the week after the assignment was posted online[1]. Please hand in your homework at the beginning of the tutorial or bring it to the lecture on Thursday morning. You can work on and submit your homework in groups of two. Please staple your pages and write your names and matriculation numbers on the first page.

Problem 37 (20 pts.)

Chasing Definitions...
Let $K = \mathbb{Q}(\sqrt[3]{2})$ and let $X = \mathrm{Hom}(K, \mathbb{C}) = \{\sigma_1, \sigma_2, \sigma_3\}$, where we suppose that $\sigma_1(\sqrt[3]{2}) = \sqrt[3]{2} \in \mathbb{R}$.

1. Work out all the definitions we discussed in class for $\mathbf{C}, \mathbf{R}, \mathbf{R}_{\pm}, \mathbf{R}_{+}^{\times}$, and $\mathbf{H}$ as *explicit as possible* in this case.

2. We equipped $\mathbf{R}$ with the inner product $\langle \cdot, \cdot \rangle$ in class. Explicitly write down an isometry to $\mathbb{R}^3$ with an appropriate inner product $(\cdot, \cdot)$ on $\mathbb{R}^3$.

3. Recall that the ring of integers $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{2}]$ (you can use this without proof) defines a lattice in $\mathbf{R}$ via $x \mapsto (\sigma_1(x), \sigma_2(x), \sigma_3(x))$, for which we also just write $\mathcal{O}_K$. Work out the image of this map in $\mathbb{R}^3$ under the identification above and then explicitly write out the theta function

$$\theta(\mathcal{O}_K, \tau) = \sum_{a \in \mathcal{O}_K} e^{\pi i \langle a \frac{\tau}{|d_K|^{1/3}}, a \rangle}$$

   in this case (along the lines of what we did in class for an imaginary quadratic field). Note: Choose a basis of $\mathcal{O}_K$, e.g. $1, \sqrt[3]{2}, \sqrt[3]{2}^2$ and figure out the image under $K \to \mathbf{R} \to \mathbb{R}^3$. Then rewrite the sum as a sum over $(n_1, n_2, n_3) \in \mathbb{Z}^3$ and write out the inner product in the exponent explicitly in terms of (n_1, n_2, n_3). You can use without proof that $d_K = -108$.

4. In this case, can you explicitly work out the function $f(t)$ that we defined in class, which statisfies

$$Z_K(2s) = L(f, s),$$

   where $Z_K(2s)$ is the completed Dedekind Zeta function.

   You can use the following facts without proof: 1) Dirichlet's unit theorem states that in this case

$$\mathcal{O}_K^{\times} = \{\pm 1\} \times \{\varepsilon^n \mid n \in \mathbb{Z}\},$$

   where $\varepsilon = \sqrt[3]{2} - 1$ is a fundamental unit. 2) The class number of K is $h_K = 1$ and hence $\zeta_K(s) = \zeta(A, s)$ for $A = [\mathcal{O}_K]$, the class of the ring of integers.

[1] This assignment is due Thursday, 23.01.20.

Problem 38 (10 pts.)

Now let $\mathbf{C}$ and $\mathbf{R}$ be as defined in class for any finite set X with an involution $\sigma \mapsto \bar{\sigma}$ for $\sigma \in X$.

1. Show that the Fourier transform (normalized as in class) of

$$f(x) = f(a, b, x) = e^{-\pi\langle a+x, a+x\rangle + 2\pi i\langle b, x\rangle}$$

   with respect to $x \in \mathbf{R}$ equals

$$\hat{f}(y) = e^{2\pi i\langle a,b\rangle} f(-b, a, y).$$

2. For $t \in \mathbf{R}_+^\times$, $a, b \in \mathbf{R}$ and $\alpha := at, \beta := bt^{-1}$, we let

$$f(x) = \varphi_t(a, b, x) = f(\alpha, \beta, tx).$$

   where $f = f_0$ is defined as in 1. Show that

$$\hat{f}(y) = \frac{1}{N(t) e^{2\pi i\langle a,b\rangle}} \varphi_{t^{-1}}(-\beta, \alpha, y),$$

   which was left to show the theta transformation law in class (for $p = 0$ for simplicity).