Homework 11

The homeworks are due on the Thursday of the week after the assignment was posted online[1]. Please hand in your homework at the beginning of the tutorial or bring it to the lecture on Thursday morning. You can work on and submit your homework in groups of two. Please staple your pages and write your names and matriculation numbers on the first page.

Problem 31 (10 pts.)

Recall that the *Bernoulli numbers*[2] B_n are defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.$$

Fill in the details left from the proof of Theorem 3.8 by showing that:

(i) $B_n = 0$ for $n \geq 2$ odd;

(ii) $(-1)^n B_n = \sum\limits_{j=0}^{n} B_j \binom{n}{j}$;

(iii) $\zeta(2n) = \frac{(-1)^{n-1} 2^{2n-1} B_{2n}}{(2n)!} \pi^{2n}$ for $n \in \mathbb{N}$.

Solution. Part (i) follows from noting that

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} - \sum_{n=0}^{\infty} B_n \frac{(-t)^n}{n!} = \frac{t}{e^t - 1} - \frac{-t}{e^{-t} - 1} = -t.$$

There are several ways to prove part (ii), one of which is to set $x = 1$ in Problem 32, part (ii) and to notice that by Problem 32, part (ii), we have $B_n(1) = B_n(0)$ for any $n \geq 2$. Part (iii) is an easy consequence of the functional equation satisfied by ζ (Corollary 3.1) and Theorem 3.8 from the Script. □

Problem 32 (10 pts.)

The *Bernoulli polynomials* $B_n(x)$ are defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$

so that $B_n = B_n(0)$. Show that:

[1] This assignment is due Thursday, 09.01.20.

[2] Note that there are two definitions in the literature. Sometimes (as in Neukirch), $\frac{te^t}{e^t-1}$ is used, which only changes the definition of B_1. If the definition from Neukirch is used, x has to be replaced by $1 + x$ in the definition of the Bernoulli polynomials in Problem 32.

(i) $B_m(x) = \sum_{n=0}^{m} \binom{m}{n} B_{m-n} x^n$;

(ii) $B_n(x+1) - B_n(x) = nx^{n-1}$;

(iii) $B_{n+1}(k+1) - B_{n+1}(1) = (n+1)(1^n + 2^n + \cdots + k^n)$.

Solution. Equating coefficients in

$$\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n = \frac{t}{e^t - 1} e^{xt} = \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} t^n \right) \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} t^n \right)$$

gives

$$\frac{B_n(x)}{n!} = \sum_{k=0}^{n} \frac{B_k}{k!} \frac{x^{n-k}}{(n-k)!}.$$

Part (ii) follows on noting that

$$t\frac{e^{(x+1)t}}{e^t - 1} - t\frac{e^{xt}}{e^t - 1} = te^{xt},$$

from where we have

$$\sum_{n=0}^{\infty} \frac{B_n(x+1) - B_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} t^{n+1}.$$

Summing up the identity (ii) for $x = 1, \ldots, k$ gives (iii). □

Problem 33 (10 pts.)

If χ is an odd primitive Dirichlet character modulo m, then

$$\sum_{a=1}^{m} \chi(a) a \neq 0.$$

Solution. Recall that the *generalized Bernoulli numbers* $B_{k,\chi}$ are defined by the formula

$$\sum_{a=1}^{m} \chi(a) \frac{te^{at}}{e^{mt} - 1} = \sum_{k=0}^{\infty} B_{k,\chi} \frac{t^k}{k!}.$$

The *Bernoulli polynomials* $B_{k,\chi}(x)$ associated to χ are defined by

$$\sum_{a=1}^{m} \chi(a) \frac{te^{(a+x)t}}{e^{mt} - 1} = \sum_{k=0}^{\infty} B_{k,\chi}(x) \frac{t^k}{k!},$$

so that $B_{k,\chi}(0) = B_{k,\chi}$ and

$$B_{1,\chi} = \frac{1}{m} \sum_{a=1}^{m} \chi(a) a.$$

As seen in class (alternatively, this is Corollary 2.10 from Ch. 7.2 in Neukirch), we have, in case χ is odd,

$$L(\chi, 1) = \frac{\pi i \tau(\chi)}{m} B_{1,\overline{\chi}}.$$

However, one of the central results in the theory of Dirichlet series says that $L(\chi, 1) \neq 0$, from where the conclusion follows. □