Homework 11

The homeworks are due on the Thursday of the week after the assignment was posted online[1]. Please hand in your homework at the beginning of the tutorial or bring it to the lecture on Thursday morning. You can work on and submit your homework in groups of two. Please staple your pages and write your names and matriculation numbers on the first page.

Problem 31 (10 pts.)

Recall that the *Bernoulli numbers*[2] B_n are defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.$$

Fill in the details left from the proof of Theorem 3.8 by showing that:

(i) $B_n = 0$ for $n \geq 2$ odd;

(ii) $(-1)^n B_n = \sum_{j=0}^{n} B_j \binom{n}{j}$;

(iii) $\zeta(2n) = \frac{(-1)^{n-1} 2^{2n-1} B_{2n}}{(2n)!} \pi^{2n}$ for $n \in \mathbb{N}$.

Problem 32 (10 pts.)

The *Bernoulli polynomials* $B_n(x)$ are defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$

so that $B_n = B_n(0)$. Show that:

(i) $B_m(x) = \sum_{n=0}^{m} \binom{m}{n} B_{m-n} x^n$;

(ii) $B_n(x+1) - B_n(x) = nx^{n-1}$;

(iii) $B_{n+1}(k+1) - B_{n+1}(1) = (n+1)(1^n + 2^n + \cdots + k^n)$.

Problem 33 (10 pts.)

If χ is an odd primitive Dirichlet character modulo m, then

$$\sum_{a=1}^{m} \chi(a) a \neq 0.$$

[1] This assignment is due Thursday, 09.01.20.

[2] Note that there are two definitions in the literature. Sometimes (as in Neukirch), $\frac{te^t}{e^t-1}$ is used, which only changes the definition of B_1. If the definition from Neukirch is used, x has to be replaced by $1 + x$ in the definition of the Bernoulli polynomials in Problem 32.