Homework 10

The homeworks are due on the Thursday of the week after the assignment was posted online[1]. Please hand in your homework at the beginning of the tutorial or bring it to the lecture on Thursday morning. You can work on and submit your homework in groups of two. Please staple your pages and write your names and matriculation numbers on the first page.

Problem 28 (10 pts.)

Show that the Fourier transform of a Schwartz function on $\mathbb{R}$ is a Schwartz function.

Solution. The idea is to first prove that $\hat{f}$ is bounded, which is easy, as

$$|\hat{f}(y)| = \left| \int_{\mathbb{R}} f(x) e^{-2\pi i x y} \, dx \right| \leq \int_{\mathbb{R}} |f(x)| \, dx \leq \int_{-1}^{1} |f(x)| \, dx + \int_{1}^{\infty} \frac{C}{x^2} \, dx = \text{const.}$$

Now the conclusion follows immediately, since the expression

$$y^k \left(\frac{d}{dy} \right)^{\ell} \hat{f}(y)$$

is bounded, as it is, by Problem 30, the Fourier transform of

$$\frac{1}{(2\pi i)^k} \left(\frac{d}{dx} \right)^k [(-2\pi i x)^{\ell} f(x)]. \qquad \square$$

Problem 29 (10 pts.)

Let a, b be positive real numbers. Show that the Mellin transforms of the functions $f(y)$ and $g(y) = f(ay^b)$ satisfy

$$L(f, s/b) = b a^{s/b} L(g, s).$$

Solution. Follows easily by a change of variables. $\square$

Problem 30 (10 pts.)

Let $f \in \mathcal{S}(\mathbb{R})$ be a Schwartz function.

1. Show that for $g(x) = f(ax)$, with $a > 0$, we have

$$\hat{g}(x) = \frac{1}{a} \hat{f} \left(\frac{x}{a} \right).$$

[1] This assignment is due Thursday, 19.12.19.

2. Show that for $g(x) = f(x-a)$ we have

$$\hat{g}(x) = e^{2\pi iax}\hat{f}(x).$$

3. Show that for $g(x) = e^{2\pi iax}f(x)$ we have

$$\hat{g}(x) = \hat{f}(x+a).$$

4. Show that for $g(x) = \frac{d}{dx}f(x)$ we have

$$\hat{g}(x) = 2\pi ix\hat{f}(x).$$

Solution. This is again easy: parts 1–3 follow just by the definition of $\hat{f}$ and by a change of variables, and part 4 by integration by parts. In particular, what is important to notice, is that, up to some factor of $2\pi i$, the Fourier transform interchages differentiation and multplication by x. This gives a very easy proof for Problem 28. □

Remark: Please note the corrected signs in the statement, as compared with the original version of this sheet. *The following exercises will be discussed in the tutorial and you do not need to hand in solutions for them.*

Exercise 28

Recall the definition of the Gamma function:

$$\Gamma(s) = \int_0^\infty e^{-t}t^s\frac{dt}{t}.$$

1. Show that the integral defining the Gamma function converges absolutely for $\mathrm{Re}(s) > 0$ and defines a holomorphic function in this half-plane. **Hint:** split the integral at $t = 1$ into an upper and a lower part.

2. Show that the Gamma function satisfies the functional equation $\Gamma(s+1) = s\Gamma(s)$.

3. Show that $\Gamma(1) = 1$.

4. Conclude that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$.

5. Let $n \in \mathbb{N}$. Use the functional equation to continue the $\Gamma(s)$ to $\mathrm{Re}(s) > -(n+1)$ and $s \neq 0, -1, -2, \ldots, -n$.

Solution. By noting that $|t^s| = t^{\mathrm{Re}(s)}$, we see that $|\Gamma(x+iy)| \leq \Gamma(x)$, so it is enough to prove the statement for $s > 0$. Split

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt = \int_0^1 e^{-t}t^{s-1}dt + \int_1^\infty e^{-t}t^{s-1}dt = I_1 + I_2.$$

We have

$$0 \leq t \leq 1 \Rightarrow e^{-t}t^{s-1} \leq t^{s-1} \Rightarrow I_1 \leq \int_0^1 t^{s-1}dt = \left.\frac{t^s}{s}\right|_0^1 = \frac{1}{s},$$

and

$$1 < t \Rightarrow e^{-t}t^{s-1} \leq Ce^{-t/2} \Rightarrow I_2 \leq C\int_1^\infty e^{-t/2}dt = \left.(-2Ce^{-t/2})\right|_1^\infty = 2Ce^{-1/2},$$

which proves the absolute convergence of $\Gamma(s)$. Integration by parts gives

$$\Gamma(s+1) = \int_0^\infty e^{-t} t^s \, dt = (-e^{-t} t^s)\big|_0^\infty + \int_0^\infty e^{-t} s t^{s-1} \, dt = s\Gamma(s).$$

Easily we compute

$$\Gamma(1) = \int_0^\infty e^{-t} \, dt = (-e^{-t})\big|_0^\infty = 1,$$

from where $\Gamma(n+1) = n!$ follows immediately. By the functional equation we get

$$\Gamma(s) = \frac{\Gamma(s+n+1)}{s(s+1)\cdots(s+n)}.$$

The right-hand side is defined and meromorphic for $\mathrm{Re}(s) > -(n+1)$, with simple poles at $s = 0, -1, \ldots, -n$. By the principle of analytic continuation, the meromorphic extension is unique and it satisfies the same functional equation. In fact, we can meromorphically continue $\Gamma(s)$ to the entire complex plane, with simple poles at $s = 0, -1, -2, \ldots$. $\square$