Homework 10

The homeworks are due on the Thursday of the week after the assignment was posted online[1]. Please hand in your homework at the beginning of the tutorial or bring it to the lecture on Thursday morning. You can work on and submit your homework in groups of two. Please staple your pages and write your names and matriculation numbers on the first page.

Problem 28 (10 pts.)

Show that the Fourier transform of a Schwartz function on $\mathbb{R}$ is a Schwartz function.
Hint: You might want to use Problem 30.

Problem 29 (10 pts.)

Let a, b be positive real numbers. Show that the Mellin transforms of the functions $f(y)$ and $g(y) = f(ay^b)$ satisfy

$$L(f, s/b) = ba^{s/b} L(g, s).$$

Problem 30 (10 pts.)

Let $f \in \mathcal{S}(\mathbb{R})$ be a Schwartz function.

1. Show that for $g(x) = f(ax)$, with $a > 0$, we have
$$\hat{g}(x) = \frac{1}{a}\hat{f}\left(\frac{x}{a}\right).$$
2. Show that for $g(x) = f(x - a)$ we have
$$\hat{g}(x) = e^{-2\pi iax}\hat{f}(x).$$
3. Show that for $g(x) = e^{-2\pi iax}f(x)$ we have
$$\hat{g}(x) = \hat{f}(x + a).$$
4. Show that for $g(x) = \frac{d}{dx}f(x)$ we have
$$\hat{g}(x) = -2\pi ix\hat{f}(x).$$

[1] This assignment is due Thursday, 19.12.19.

The following exercises will be discussed in the tutorial and you do not need to hand in solutions for them.

Exercise 28

Recall the definition of the Gamma function:

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}.$$

1. Show that the integral defining the Gamma function converges absolutely for $\mathrm{Re}(s) > 0$ and defines a holomorphic function in this half-plane. **Hint:** split the integral at $t = 1$ into an upper and a lower part.
2. Show that the Gamma function satisfies the functional equation $\Gamma(s+1) = s\Gamma(s)$.
3. Show that $\Gamma(1) = 1$.
4. Conclude that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$.
5. Let $n \in \mathbb{N}$. Use the functional equation to continue the $\Gamma(s)$ to $\mathrm{Re}(s) > -(n+1)$ and $s \neq 0, -1, -2, \ldots, -n$.