Static solutions to the spherically symmetric Einstein-Vlasov system: a particle-number-Casimir approach

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Abstract

Existence of spherically symmetric solutions to the Einstein-Vlasov system is well-known. However, it is an open problem whether or not static solutions arise as minimizers of a variational problem. Apart from being of interest in its own right, it is the connection to non-linear stability that gives this topic its importance. This problem was considered in [28], but as has been pointed out in [5], the paper [28] contained serious flaws. In this work we construct static solutions by solving the Euler-Lagrange equation for the energy density ρ as a fixed point problem. The Euler-Lagrange equation originates from the particle number-Casimir functional introduced in [28]. We then define a density function f on phase space which induces the energy density ρ and we show that it constitutes a static solution of the Einstein-Vlasov system. Hence we settle rigorously parts of what the author of [28] attempted to prove.

1 Introduction

The first proof of existence of spherically symmetric static solutions to the Einstein-Vlasov system was given by Rein and Rendall in 1993, see [26]. Several simplifications and generalizations have since then been obtained, and we refer to [2] for a review. By now, existence of a wide class of static solutions has been established, including a proof in the massless case [4], which requires very different techniques. The question whether or not static solutions arise as minimizers of a variational problem has, on the other hand, remained open.

The aim of the work [28] was to settle this question. However, it was shown in [5] that there are serious errors in [28], which left the problem unsolved. In mathematical terms, the issue is if there are static solutions to the spherically symmetric Einstein-Vlasov system that are minimizers to the particle number-Casimir functional

$$\mathcal{D}(f) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{\lambda_f}\,(\hat{\Phi}(f) - \alpha f)\;dx\,dv =: \int_{\mathbb{R}^3} e^{\lambda_f}\,H(f)\,dx, \tag{1.1}$$

as introduced by Wolansky in [28]. Here f is the density function on phase space, λ_f is a metric function that depends on f, and $\hat{\Phi}$ is the Legendre transform of a given ansatz function Φ; lastly, $\alpha > 0$ is a constant. We have denoted by H the part of the functional $\mathcal{D}$ for which the integration is taken over momentum space.

For a motivation and background concerning the form of the functional $\mathcal{D}$ we refer to [28]. The first goal is to find a minimizer of $\mathcal{D}$ under a mass constraint, i.e., it is required that $m(r) \leq M$, for some $M > 0$, where

$$m(r) := 4\pi \int_0^r s^2 \rho(s) ds,$$

and where ρ is given in terms of f, see (2.5) below. The second goal is to show that the minimizer is a static solution to the Einstein-Vlasov system. The route proposed in [28] is in fact somewhat different and another variational problem is considered. Indeed, in this paper a certain Lagrangian $L = L(r, q, p)$ is defined and the following functional is introduced:

$$\mathcal{L}(m) = \int_0^\infty L(r, m(r), m'(r))\,dr. \tag{1.2}$$

The variational problem is then to find a minimizer m of $\mathcal{L}$ under the condition that $m(r) \leq M$. This problem is closely related to the former, as is shown in Section 6 below. If a minimizer m could be obtained, it would then be necessary to prove that it satisfies the corresponding Euler-Lagrange equation. However, for doing this a major difficulty arises. It is related to the fact that the functional is not bounded from below: $|\mathcal{L}|$ becomes arbitrary large for configurations that are on the verge to admitting trapped surfaces. Therefore a further condition on the functions is needed to avoid such configurations in order to obtain a lower bound. Such a condition, however, drastically complicates the optimization problem, since it introduces additional "boundaries", resulting in the fact that the minimization problem is turned into an obstacle problem. This is one of the reasons for the gaps in [28], where an additional "barrier condition" on the mass function had been added to the set.

The Euler-Lagrange equation mentioned above can be formulated as a fixed point equation for the energy density ρ. The question we are going to address in this work is if there exists a solution ρ_* to the Euler-Lagrange equation, and if it then is possible to define a density function f_* that induces the energy density ρ_*, and that constitutes a static solution of the Einstein-Vlasov system. We give an affirmative answer to this question. Our result rigorously settles what the author of [28] attempted to prove, and it provides a connection between a static solution and the variational problem for the density function. In particular, f_* is a minimizer of the functional H as introduced above, cf. Section 5 for a precise meaning. For a complete understanding, it nevertheless remains to be shown that (under reasonable constraints) there exists a minimizer to the full particle number-Casimir functional $\mathcal{D}$, which constitutes a static solution of the Einstein-Vlasov system.

In the literature there is another functional that has been considered in this context, which is called the energy-Casimir functional, cf. [16]. It has been shown in this paper that steady states of the Einstein-Vlasov system are solutions to the Euler-Lagrange equations that correspond to the energy-Casimir functional; also see [18]. In addition, the positive definiteness of the second variation of the energy-Casimir functional at steady states holds [16, 17], and this is central for the proofs of linear stability and instability in these works. Therefore the question arises how the particle number-Casimir functional compares to the energy-Casimir functional. Since this paper is the first rigorous instance where the former is used, the following reasoning contains quite a bit of speculation. A variational problem can certainly be formulated for the energy-Casimir functional as well, but there is no result in this case either that guarantees the existence of a minimizer. There is also one difference between the two variational problems that is worth mentioning. For the particle number-Casimir functional, the minimization is done under a mass constraint as described above, whereas the analogous constraint for the energy-Casimir functional is the particle number. If one considers the variational problems numerically, it is more convenient to use the mass constraint rather than the particle number constraint. The reason is that the expression for the latter contains the metric function λ (in the present coordinates). Hence it is a subtle issue to vary f such that the particle number remains unchanged, whereas it is rather straightforward to vary f while keeping the mass fixed. Numerical insights are generally helpful in order to make analytic progress, and from this perspective the particle number-Casimir functional could be advantageous, as compared to the energy-Casimir functional.

Apart from being of interest in its own right, it is the connection to non-linear stability that gives the topic of steady states as minimizers its importance. In the case of the Vlasov-Poisson system (that is the Newtonian analogue of the Einstein-Vlasov system), it is well-known that a large class of static solutions (steady states) can be obtained as minimizers of an energy-Casimir functional, cf. [25] and the references therein. This fact has been central for proving non-linear stability of steady states of the system, in the approach taken by Guo and Rein in 1999, see [14]. The authors considered spherically symmetric steady states, and the admissible class of perturbations also consisted of spherically symmetric functions. Since then, this result has been improved, and so far the most general conditions could be treated by Lemou, Méhats and Raphaël in 2012, cf. [19], where the class of perturbations is general and not restricted to spherical symmetry. For a review of this topic we refer to [22, 25].

In contrast to the Newtonian case, the non-linear stability problem for static solutions - or steady states - of the spherically symmetric Einstein-Vlasov system is open. As has been noted above, linear stability and instability has been shown in [16, 17]. In the Newtonian case it holds that any steady state for which the density function is non-increasing with respect to the particle energy is stable. For relativistic steady states, under the same condition there is numerical and analytic evidence that these can be both stable or unstable, cf. [7, 13, 16, 17]. The main quantities that determine the stability properties in the relativistic case seem to be the central redshift and the binding energy. The understanding of non-linear stability is thus quite different in the two cases.

One reason for this difference is the possibility of black hole formation in the relativistic case, which is in stark contrast to the Vlasov-Poisson system, where solutions are known to exist globally in time, independent of the size of the initial data; cf. [23, 21]. For the Einstein-Vlasov system on the other hand, there exist initial data that lead to black holes in the evolution, see [10, 6, 3]. If this happens, the spacetime will be geodesically incomplete. Let us here mention that even if black holes form, the solutions may still be global in (coordinate) time. For instance, in so called Schwarzschild coordinates, it was shown in [6] that there are initial data that lead to the formation of black holes, where the solutions exist globally in time. For initial data close to Minkowski space, global existence and stability has been shown in [11, 20]. The possibility of black hole formation in general relativity influences the non-linear stability problem, and as discussed above, it influences the formulation of the variational problem. In order to get a meaningful variational problem, one has to put constraints on the function space over which the minimization is carried out, since otherwise the functional has no lower bound. Hence, even if a minimizer can be shown to exist, it is yet necessary to control certain quantities in the evolution to guarantee that the density function stays in the function space, and in particular one has to show that black holes cannot form. If such a control can be achieved (for instance by means of a good stability estimate), then the information provided by the variational approach could be used. For the variational problem, a direct approach by means of minimizing sequences is notoriously difficult, since one has to deal with non-compactness both in v-space and in x-space, and no straightforward concentration compactness argument is available. And certainly there is no guarantee that a variational approach, if successful, would give an advantage, as compared to the method of [16]. Simply stated, at the current time the understanding of non-linear stability is still shrouded in mystery.

Let us finally stress that the results in this work do not lead to the existence of a minimizer of the variational problem and therefore there is no direct link between our result and stability. However, our work provides insights about important tools that are useful for investigating this problem further, and moreover we get the corresponding arguments in [28] straight, which we consider to be our main contribution.

The outline of the paper is as follows. In the next section we introduce the Einstein-Vlasov system. In Section 3 we derive the Euler-Lagrange equation as discussed above and we state our main results, Theorems 3.3 and 3.4. In Section 4 we formulate the Euler-Lagrange equation as a fixed point problem for the energy density ρ and we show existence of solutions to this equation. Having obtained a solution ρ_* to the fixed point equation, we define in Section 5 a density function f_* that induces ρ_* and that constitutes a static solution of the Einstein-Vlasov system. In Section 6 the relation to stability is discussed. Finally, in Section 7 (which is an appendix) we collect properties of the Legendre transform and of some particular functions that are crucial for the argument.

2 The Einstein-Vlasov system

Below we use units such that $G = 1$ and $c = 1$, where G is the gravitational constant and c is the speed of light. For a function $g = g(t, r)$ we sometimes use the notation $g' := \partial_r g$ and $\dot{g} := \partial_t g$.

The metric of a spherically symmetric spacetime takes the following form in Schwarzschild coordinates:

$$ds^2 = -e^{2\mu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2),$$

where $r \geq 0$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$ and $t \in \mathbb{R}$. To ensure asymptotic flatness and a regular center, the following boundary conditions are imposed:

$$\lim_{r\to\infty} \lambda(t, r) = \lim_{r\to\infty} \mu(t, r) = 0 = \lambda(t, 0).$$

We will now formulate the spherically symmetric Einstein-Vlasov system. We refer to [2, 24, 27] for more information about this system and its derivation. The fundamental quantity that describes matter within the model is the density function f, which is defined on phase space. In this work we use two different coordinate systems on phase space which are standard in the literature, cf. e.g. [24]: either we write $f = f(t, r, w, l^2)$ or $f = f(x, v) = f(x_1, x_2, x_3, v_1, v_2, v_3)$, where these coordinates are related by

$$x = r(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),$$
$$w = \frac{x \cdot v}{r},\ \ \ell^2 = |x \wedge v|^2.$$

The variables w and l^2 can be thought of as the momentum in the radial direction and the square of the angular momentum, respectively. Sometimes we will also use the notation

$$\beta := \ell^2.$$

For later reference we note that $dx = 4\pi r^2\, dr$, and that

$$dx\, dv = 8\pi^2\, dr\, d\ell\, \ell\, dw = 4\pi^2\, dr\, d\ell^2\, dw. \tag{2.1}$$

The Einstein-Vlasov system is given by the Einstein equations

$$e^{-2\lambda}(2r\lambda' - 1) + 1 = 8\pi r^2\, \rho, \tag{2.2}$$

$$e^{-2\lambda}(2r\mu' + 1) - 1 = 8\pi r^2 p, \tag{2.3}$$
$$\dot{\lambda} = -4\pi r\, e^{\lambda+\mu}\, j,$$
$$e^{-2\lambda}\Big(\mu'' + (\mu' - \lambda')\Big(\mu' + \frac{1}{r}\Big)\Big) - e^{-2\mu}\Big(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\mu})\Big) = 8\pi\, p_T,$$

together with the Vlasov equation

$$\partial_t f + e^{\mu-\lambda} \frac{w}{\sqrt{1 + w^2 + \ell^2/r^2}} \partial_r f - \Big(\dot{\lambda}\, w + e^{\mu-\lambda}\mu' \sqrt{1 + w^2 + \ell^2/r^2} - e^{\mu-\lambda} \frac{\ell^2}{r^3\sqrt{1 + w^2 + \ell^2/r^2}}\Big)\, \partial_w f = 0. \tag{2.4}$$

Here the non-vanishing components of the energy-momentum tensor are defined by

$$\rho(t, r) = \frac{\pi}{r^2} \int_{\mathbb{R}} \int_0^\infty \sqrt{1 + w^2 + \ell^2/r^2}\, f(t, r, w, \ell^2)\, d\ell^2\, dw, \tag{2.5}$$

$$p(t, r) = \frac{\pi}{r^2} \int_{\mathbb{R}} \int_0^\infty \frac{w^2}{\sqrt{1 + w^2 + \ell^2/r^2}} f(t, r, w, \ell^2)\, d\ell^2\, dw, \tag{2.6}$$

$$p_T(t, r) = \frac{\pi}{2r^4} \int_{\mathbb{R}} \int_0^\infty \frac{\ell^2}{\sqrt{1 + w^2 + \ell^2/r^2}} f(t, r, w, \ell^2)\, d\ell^2\, dw, \tag{2.7}$$

$$j(t, r) = \frac{\pi}{r^2} \int_{\mathbb{R}} \int_0^\infty w\, f(t, r, w, \ell^2)\, d\ell^2\, dw. \tag{2.8}$$

The quantities ρ, p, p_T and j are the energy density, the radial pressure, the tangential pressure and the current, respectively. The equations above are not independent and typically one considers the reduced system (2.2)-(2.3) together with (2.4) and (2.5)-(2.6). It is straightforward to show that a solution to the reduced system yields a solution to the full system, cf. [24]. We will mainly consider static solutions in this work, and in the present coordinates such solutions are simply time independent.

In the proofs we will use a couple of well-known consequences of the Einstein equations. First we note that equation (2.2) can be integrated to give

$$e^{-2\lambda(t,r)} = 1 - \frac{2m(t, r)}{r}, \tag{2.9}$$

where the mass function m is defined by

$$m(t, r) = 4\pi \int_0^r s^2 \rho(t, s)\, ds. \tag{2.10}$$

Moreover, equation (2.3) can be written in terms of m and λ as

$$\mu(t, r) = -\int_r^\infty e^{2\lambda(t,s)} \Big(\frac{m(t, s)}{s^2} + 4\pi s\, p(t, s)\Big)\, ds. \tag{2.11}$$

If a density function f is given we will denote by ρ_f, p_f, m_f, λ_f and μ_f the functions defined by (2.5), (2.6), (2.10), (2.9) and (2.11), respectively.

3 Preliminaries and main results

Some of the quantities below already appeared in [28]. Let $\phi(s) = s_+^\sigma$ for $\sigma \in]0, \frac{3}{2}[$ and $s_+ = \max\{s, 0\}$. Define $\Phi(s) = \frac{1}{\sigma+1}\, s_+^{\sigma+1}$. Then the Legendre transform of Φ is calculated to be

$$\hat{\Phi}(u) = \begin{cases} \frac{\sigma}{\sigma+1}\, u^{1+1/\sigma} & : \quad u \in [0, \infty[\\ \infty & : \quad u \in]-\infty, 0[\end{cases} ; \tag{3.1}$$

a few facts concerning Legendre transforms are recalled in Section 7.2. Next we define

$$\Psi(\varepsilon, r) = \int_0^\infty d\beta \int_{\mathbb{R}} dw\, \Phi(\alpha - \varepsilon\sqrt{1 + w^2 + \beta/r^2}) \tag{3.2}$$

for a fixed $\alpha > 0$. We also let

$$G(\varepsilon) = \int_0^\infty \xi^2\, (\alpha - \varepsilon\sqrt{1+\xi^2})_+^{\sigma+1}\, d\xi, \quad \varepsilon \in \mathbb{R}. \tag{3.3}$$

Using the change of variables $(w, y) = \xi(\cos\theta, \sin\theta)$, $|\det d(w, y)/d(\theta, \xi)| = \xi$, it is found that

$$\begin{aligned} \Psi(\varepsilon, r) &= \frac{1}{\sigma+1} \int_0^\infty d\beta \int_{\mathbb{R}} dw\, (\alpha - \varepsilon\sqrt{1 + w^2 + \beta/r^2})_+^{\sigma+1} \\ &= \frac{2r^2}{\sigma+1} \int_0^\infty dy\, y \int_{\mathbb{R}} dw\, (\alpha - \varepsilon\sqrt{1 + w^2 + y^2})_+^{\sigma+1} \\ &= \frac{2r^2}{\sigma+1} \int_0^\pi d\theta \sin\theta \int_0^\infty d\xi\, \xi^2\, (\alpha - \varepsilon\sqrt{1+\xi^2})_+^{\sigma+1} \\ &= \frac{4r^2}{\sigma+1}\, G(\varepsilon) \end{aligned} \tag{3.4}$$

for G from (3.3). Another way to represent Ψ is as follows. We have

$$\begin{aligned} \Psi(\varepsilon, r) &= \int_0^\infty d\beta \int_{\mathbb{R}} dw\, \frac{d}{dw}\, w\, \Phi(\alpha - \varepsilon\sqrt{1 + w^2 + \beta/r^2}) \\ &= \int_0^\infty d\beta \int_{\mathbb{R}} dw\, \Big[w\, \Phi(\alpha - \varepsilon\sqrt{1 + w^2 + \beta/r^2})\Big]_{w=-\infty}^{w=\infty} \\ &\quad + \varepsilon \int_0^\infty d\beta \int_{\mathbb{R}} dw\, \frac{w^2}{\sqrt{1 + w^2 + \beta/r^2}}\, \Phi'(\alpha - \varepsilon\sqrt{1 + w^2 + \beta/r^2}) \\ &= \varepsilon \int_0^\infty d\beta \int_{\mathbb{R}} dw\, \frac{w^2}{\sqrt{1 + w^2 + \beta/r^2}}\, \phi(\alpha - \varepsilon\sqrt{1 + w^2 + \beta/r^2}). \end{aligned} \tag{3.5}$$

Using (3.4), we calculate

$$\hat{\Psi}(u,r) = \sup_{\varepsilon\in\mathbb{R}} \Big(\varepsilon u - \frac{4r^2}{\sigma+1}\, G(\varepsilon)\Big) = \frac{4r^2}{\sigma+1} \sup_{\varepsilon\in\mathbb{R}} \Big(\frac{\sigma+1}{4r^2}\, \varepsilon u - G(\varepsilon)\Big) = \frac{4r^2}{\sigma+1}\, \hat{G}\Big(\frac{\sigma+1}{4r^2}\, u\Big),$$

which can be re-expressed as

$$\hat{G}(u) = \frac{\sigma+1}{4r^2}\, \hat{\Psi}\Big(\frac{4r^2}{\sigma+1}\, u, r\Big). \tag{3.6}$$

Define the Lagrangian

$$L(r,q,p) = -\frac{r^{5/2}}{\sqrt{r-2q}} \inf_{\varepsilon\in\mathbb{R}} \Big(G(\varepsilon) + \varepsilon\,\kappa \frac{p}{r^2}\Big) = \frac{r^{5/2}}{\sqrt{r-2q}}\, \hat{G}\Big(-\kappa\frac{p}{r^2}\Big)$$

for $\kappa = \frac{\sigma+1}{16\pi^2}$. The derivatives are

$$\frac{\partial L}{\partial q} = \frac{r^{5/2}}{(r-2q)^{3/2}}\, \hat{G}\Big(-\kappa\frac{p}{r^2}\Big), \quad \frac{\partial L}{\partial p} = -\frac{\kappa}{\sqrt{1-\frac{2q}{r}}}\, \hat{G}'\Big(-\kappa\frac{p}{r^2}\Big). \tag{3.7}$$

Suppose now that there is a nice class of functions $\mathcal{A} \ni m$ such that

$$\mathcal{L}(m) = \int_0^\infty L(r, m(r), m'(r))\, dr \tag{3.8}$$

is minimized over $\mathcal{A}$ by some nice function $m_* \in \mathcal{A}$ such that $\lim_{r\to 0} \frac{m_*(r)}{r} = 0$, $m_*(\infty) = M$ and $\frac{2m_*(r)}{r} < 1$. Define $\rho_*(r) = \frac{1}{4\pi r^2}\, m_*'(r)$. Then $4\pi \int_0^\infty r^2\, \rho_*(r)\, dr = \int_0^\infty m_*'(r)\, dr = m_*(\infty) - m_*(0) = M$, and one would expect that the Euler-Lagrange equation

$$\frac{d}{dr}\Big[\frac{\partial L}{\partial m'}\Big] = \frac{\partial L}{\partial m}$$

is satisfied for m_*. Due to (3.7) this reads as

$$-\kappa\, \frac{d}{dr}\Big[\frac{1}{\sqrt{1-\frac{2m_*(r)}{r}}}\, \hat{G}'\Big(-\kappa\frac{m_*'(r)}{r^2}\Big)\Big] = \frac{r^{5/2}}{(r-2m_*(r))^{3/2}}\, \hat{G}\Big(-\kappa\frac{m_*'(r)}{r^2}\Big). \tag{3.9}$$

In the sequel we will study (3.9) in some more detail.

Remark 3.1 Let $\mathcal{G}(s) = \hat{G}(-s)$ and, for an appropriate function $m(r) = 4\pi \int_0^r s^2 \rho(s)\, ds$ such that $\frac{2m(r)}{r} < 1$, put $\zeta(r) = \mathcal{G}'(\kappa\frac{m'(r)}{r^2}) = \mathcal{G}'(4\pi\kappa\rho(r))$. Also define

$$l(r) = \kappa\, \frac{d}{dr}\Big[\frac{1}{\sqrt{1-\frac{2m(r)}{r}}}\, \hat{G}'\Big(-\kappa\frac{m'(r)}{r^2}\Big)\Big] + \frac{r^{5/2}}{(r-2m(r))^{3/2}}\, \hat{G}\Big(-\kappa\frac{m'(r)}{r^2}\Big),$$

so that m satisfies the Euler-Lagrange equations (3.9) on some interval $I \subset]0,\infty[$ iff $l(r) = 0$ for $r \in I$. Then l can be rewritten as

$$l(r) = \frac{1}{\sqrt{1-\frac{2m(r)}{r}}}\Big(-\kappa\,\zeta'(r) + \kappa\,\frac{m(r)}{r^2}\,\frac{1}{1-\frac{2m(r)}{r}}\,\zeta(r) - \frac{r}{1-\frac{2m(r)}{r}}\,G(-\zeta(r))\Big), \tag{3.10}$$

cf. [28, eq. (36), p. 221] (which contains a misprint). To show this, using (7.15), we have

$$\begin{aligned}
l(r) &= -\kappa\,\frac{d}{dr}\Big[\frac{1}{\sqrt{1-\frac{2m(r)}{r}}}\,\mathcal{G}'\Big(\kappa\frac{m'(r)}{r^2}\Big)\Big] + \frac{r^{5/2}}{(r-2m(r))^{3/2}}\,\mathcal{G}\Big(\kappa\frac{m'(r)}{r^2}\Big)\\
&= -\kappa\,\frac{1}{\sqrt{1-\frac{2m(r)}{r}}}\,\frac{d}{dr}\Big[\mathcal{G}'\Big(\kappa\frac{m'(r)}{r^2}\Big)\Big] + \kappa\,\frac{1}{(1-\frac{2m(r)}{r})^{3/2}}\Big(\frac{m(r)}{r^2} - 4\pi r\rho(r)\Big)\,\mathcal{G}'\Big(\kappa\frac{m'(r)}{r^2}\Big)\\
&\quad + \frac{r^{5/2}}{(r-2m(r))^{3/2}}\,\mathcal{G}\Big(\kappa\frac{m'(r)}{r^2}\Big)\\
&= \frac{1}{\sqrt{1-\frac{2m(r)}{r}}}\Big(-\kappa\,\zeta'(r) + \kappa\,\frac{1}{1-\frac{2m(r)}{r}}\Big(\frac{m(r)}{r^2} - 4\pi r\rho(r)\Big)\,\mathcal{G}'(4\pi\kappa\rho(r))\\
&\qquad\qquad + \frac{r}{1-\frac{2m(r)}{r}}\,\mathcal{G}(4\pi\kappa\rho(r))\Big)\\
&= \frac{1}{\sqrt{1-\frac{2m(r)}{r}}}\Big(-\kappa\,\zeta'(r) + \kappa\,\frac{1}{1-\frac{2m(r)}{r}}\,\frac{m(r)}{r^2}\,\zeta(r)\\
&\qquad\qquad + \frac{r}{1-\frac{2m(r)}{r}}\Big[\mathcal{G}(4\pi\kappa\rho(r)) - 4\pi\kappa\rho(r)\,\mathcal{G}'(4\pi\kappa\rho(r))\Big]\Big)\\
&= \frac{1}{\sqrt{1-\frac{2m(r)}{r}}}\Big(-\kappa\,\zeta'(r) + \kappa\,\frac{1}{1-\frac{2m(r)}{r}}\,\frac{m(r)}{r^2}\,\zeta(r) - \frac{r}{1-\frac{2m(r)}{r}}\,G(-\mathcal{G}'(4\pi\kappa\rho(r)))\Big),
\end{aligned}$$

which yields (3.10).

As a corollary, we note the following fact: if $l(r) = 0$ for r in some interval $I \subset]0,\infty[$, then ρ is strictly decreasing on I; see [28, p. 221]. Indeed, from (3.10) we get

$$\zeta'(r) = \frac{m(r)}{r^2}\,\frac{1}{1-\frac{2m(r)}{r}}\,\zeta(r) - \kappa^{-1}\frac{r}{1-\frac{2m(r)}{r}}\,G(-\zeta(r)), \quad r \in I.$$

We have $\mathcal{G}'(s) < 0$ for $s \in [0,\infty[$ by Lemma 7.8 and $G(\varepsilon) > 0$ for $\varepsilon \in]0,\alpha[$ by Lemma 7.3. In addition, $-\zeta(r) = -\mathcal{G}'(4\pi\kappa\rho(r)) \in]0,\alpha[$, due to Lemma 7.4. Thus we obtain $\zeta'(r) < 0$ for $r \in I$. If $r,\tilde{r} \in I$ are such that $r < \tilde{r}$, then $\mathcal{G}'(4\pi\kappa\rho(r)) = \zeta(r) > \zeta(\tilde{r}) = \mathcal{G}'(4\pi\kappa\rho(\tilde{r}))$. Since $\mathcal{G}'$ is strictly increasing on $]0,\infty[$ due to Lemma 7.8, it follows that $\rho(r) > \rho(\tilde{r})$. ♢

Next we need to fix some constants. According to Lemmas 7.3, 7.4 and 7.5 in the appendix (Section 7.1), there are $c_2 > c_1 > 0$ (independent of α, see Remark 7.7) such that

$$c_1\,\alpha^{-3/2}\,(\alpha-\varepsilon)^{5/2+\sigma} \le G(\varepsilon) \le c_2\,\alpha^{-3/2}\,(\alpha-\varepsilon)^{5/2+\sigma}, \tag{3.11}$$

$$c_1\,\alpha^{-3/2}\,(\alpha-\varepsilon)^{3/2+\sigma} \le |G'(\varepsilon)| \le c_2\,\alpha^{-3/2}\,(\alpha-\varepsilon)^{3/2+\sigma}, \tag{3.12}$$

$$c_1\,\alpha^{-3/2}\,(\alpha-\varepsilon)^{1/2+\sigma} \le G''(\varepsilon) \le c_2\,\alpha^{-3/2}\,(\alpha-\varepsilon)^{1/2+\sigma}, \tag{3.13}$$

all for $\varepsilon \in [\frac{1}{\sqrt{2}}\,\alpha, \alpha]$. Due to Lemma 7.8 and (7.12) there are further constants $c_3^* > 0$ and $c_4 > c_3 > 0$ (independent of α) with the property that

$$c_3\,\alpha^{\frac{3}{3+2\sigma}}\,s^{-\frac{1+2\sigma}{3+2\sigma}} \le \mathcal{G}''(s) \le c_4\,\alpha^{\frac{3}{3+2\sigma}}\,s^{-\frac{1+2\sigma}{3+2\sigma}}, \quad s \in]0, s_0], \tag{3.14}$$

for $s_0 = 2c_3^*\alpha^\sigma$.

Lemma 3.2 *Suppose that $0 \le \rho(r) \le \eta$, where $\eta > 0$ satisfies*

$$\eta \le \min\{1, \eta_*\}, \tag{3.15}$$

for a suitable $\eta_* > 0$ that is small enough (depending on σ and α). Then

$$c_5\,\alpha^{\frac{3}{3+2\sigma}}\,\rho(r)^{\frac{2}{3+2\sigma}} \le \alpha + \zeta(r) \le c_6\,\alpha^{\frac{3}{3+2\sigma}}\,\rho(r)^{\frac{2}{3+2\sigma}} \tag{3.16}$$

for $\zeta(r) = \mathcal{G}'(4\pi\kappa\rho(r))$, defining

$$c_5 = c_3\,(4\pi\kappa)^{\frac{2}{3+2\sigma}}\,\frac{3+2\sigma}{2} \quad \text{and} \quad c_6 = c_4\,(4\pi\kappa)^{\frac{2}{3+2\sigma}}\,\frac{3+2\sigma}{2}.$$

In particular, we have

$$-\zeta(r) \in \Big[\frac{1}{\sqrt{2}}\,\alpha, \alpha\Big]. \tag{3.17}$$

Furthermore, it holds that

$$c_7\,\alpha^{\frac{3}{3+2\sigma}}\,\rho(r)^{\frac{5+2\sigma}{3+2\sigma}} \le G(-\zeta(r)) \le c_8\,\alpha^{\frac{3}{3+2\sigma}}\,\rho(r)^{\frac{5+2\sigma}{3+2\sigma}}, \tag{3.18}$$

$$c_7\,\alpha^{-\frac{3}{3+2\sigma}}\,\rho(r)^{\frac{1+2\sigma}{3+2\sigma}} \le G''(-\zeta(r)) \le c_8\,\alpha^{-\frac{3}{3+2\sigma}}\,\rho(r)^{\frac{1+2\sigma}{3+2\sigma}}, \tag{3.19}$$

for

$$c_7 = \min\{c_5^{\frac{5+2\sigma}{2}}, c_5^{\frac{1+2\sigma}{2}}\}\,c_1 \quad \text{and} \quad c_8 = \max\{c_6^{\frac{5+2\sigma}{2}}, c_6^{\frac{1+2\sigma}{2}}\}\,c_2.$$

Proof: Since $0 \le 4\pi\kappa\rho(r) \le 4\pi\kappa\eta \le s_0$, for s_0 according to (3.14), it follows from (3.14) that

$$\begin{aligned} \alpha + \zeta(r) &= \mathcal{G}'(4\pi\kappa\rho(r)) - \mathcal{G}'(0) = \int_0^{4\pi\kappa\rho(r)} \mathcal{G}''(\lambda)\,d\lambda \\ &\ge c_3\,\alpha^{\frac{3}{3+2\sigma}} \int_0^{4\pi\kappa\rho(r)} \lambda^{-\frac{1+2\sigma}{3+2\sigma}}\,d\lambda = c_5\,\alpha^{\frac{3}{3+2\sigma}}\,\rho(r)^{\frac{2}{3+2\sigma}}. \end{aligned}$$

The upper bound in (3.16) can be shown in the same way, due to the upper bound on $\mathcal{G}''(s)$ from (3.14). The lower bound for $-\zeta(r)$ in (3.17) is a consequence of (3.15) and (3.16), whereas the upper bound follow from (3.16) and the fact that $\rho \geq 0$. Next, (3.18) and (3.19) are due to (3.11) and (3.13), combined with (3.16), noticing that (3.17) holds.

For the record (i.e., for the case that in the future some of the results might be needed with different α's), we note that

$$\eta_* = \min\left\{\frac{c_3^*}{2\pi\kappa}, \frac{1}{4\pi\kappa}\left(\frac{\sqrt{2}}{\sqrt{2}-1}\right)^{-\frac{3+2\sigma}{2}}\left(\frac{3}{2}+\sigma\right)^{-\frac{3+2\sigma}{2}} c_4^{-\frac{3+2\sigma}{2}}\right\}\alpha^\sigma$$

is a suitable choice, which leads to the values of $c_5, \ldots, c_8$ as given above. □

The main results in this work can now be stated.

Theorem 3.3 *For every fixed $\eta > 0$ satisfying (3.15) and for every $\alpha > 0$ there exists a finite $R_* > 0$ and a solution $m_* \in C^2([0, R_*])$ of the Euler-Lagrange equation (3.9) such that $0 \leq \frac{2m_*(r)}{r} < 1$ for $r \in [0, R_*]$, $0 < \rho_*(r) \leq \eta$ for $r \in [0, R_*[$, $\rho_*(0) = \eta$, $\rho_*(R_*) = 0$, and ρ_* is strictly decreasing.*

The solution ρ_* induces a density function f_* that is a static solution of the Einstein-Vlasov system.

Theorem 3.4 *Let $\phi(s) = s_+^\sigma$ for $\sigma \in]0, \frac{3}{2}[$ be as above. For every $\alpha > 0$ and $\eta > 0$ sufficiently small (depending only on α) there is a static solution to the Einstein-Vlasov system of the form*

$$f_*(r, w, \beta) = \phi(\alpha - ce^{\mu(r)}\sqrt{1 + w^2 + \beta/r^2})$$

for $r \in [0, R]$, which is the support of the steady state; here $c > 0$ is a suitable parameter that will be determined in the proof. Furthermore, the associated ρ_* is a solution of the Euler-Lagrange equation (3.9).

Remark 3.5 Recall that the Euler-Lagrange equation (3.9) is associated with the functional (3.8), which in turn is closely related to the functional (1.1) as shown in Section 6. Thus our result gives support to the claim that the static solutions constructed in Theorem 3.4 are minimizers of the functional (1.1) under reasonable constraints. However, a rigorous proof is yet missing. ♢

4 Solving the Euler-Lagrange equation

Let η be such that (3.15) is verified. We consider the fixed point equation

$$\rho(r) = \eta - \frac{1}{4\pi\kappa^2}\int_0^r G''(-\zeta(s))\left[\kappa\,\frac{m(s)}{s^2}\,\frac{1}{1-\frac{2m(s)}{s}}\,(-\zeta(s)) + \frac{s}{1-\frac{2m(s)}{s}}\,G(-\zeta(s))\right]ds. \tag{4.1}$$

Note that this is not an integrated ODE in the standard form, since the dependence of m on ρ is $m(r) = 4\pi \int_0^r s^2\rho(s)\, ds$. Equation (4.1) is obtained from the equation $l(r) = 0$, where $l(r)$ is given by equation (3.10), by using the properties of the Legendre transform, as specified in Section 7.2.

We start with some auxiliary observations regarding the right-hand side of (4.1).

Lemma 4.1 *(a) Let $R > 0$ and $\rho \in C([0,R])$ be such that $\rho(r) > 0$ and $\frac{2m(r)}{r} < 1$ for $r \in [0,R]$. Define*

$$I_\rho(r) = G''(-\zeta(r)) \Big[\kappa \frac{m(r)}{r^2} \frac{1}{1-\frac{2m(r)}{r}} (-\zeta(r)) + \frac{r}{1-\frac{2m(r)}{r}} G(-\zeta(r))\Big], \quad r \in [0,R], \tag{4.2}$$

to be the integrand in (4.1), where $\frac{m(r)}{r^2}$ and $\frac{m(r)}{r}$ are taken to be zero at $r = 0$. Then $I_\rho(r) > 0$ for $r \in]0,R]$.
(b) If $\rho \in C([0,R])$ is a solution to (4.1) on some interval $[0,R]$, then in fact ρ is continuously differentiable on $[0,R]$ and satisfies $\rho'(r) = -\frac{1}{4\pi\kappa^2} I_\rho(r)$, so that $\rho'(r) < 0$ for $r \in]0,R]$.
(c) There is a constant $c_9 > 0$ with the following property. Let $R > 0$ and $\rho \in C([0,R])$ be such that $0 < \rho(r) \le \eta$ and $\frac{2m(r)}{r} \le \frac{1}{2}$ for $r \in [0,R]$. Then

$$I_\rho(r) \le c_9 (\alpha^{\frac{2\sigma}{3+2\sigma}} \eta^{\frac{4(1+\sigma)}{3+2\sigma}} + \eta^2)\, R.$$

(d) There is a constant $c_{10} > 0$ with the following property. Let $R > 0$ and $\rho, \tilde\rho \in C([0,R])$ be such that $0 < \rho(r) \le \eta$, $0 < \tilde\rho(r) \le \eta$ and $\frac{2m(r)}{r} < 1$, $\frac{2\tilde m(r)}{r} < 1$ for $r \in [0,R]$. Then one can find a constant $\hat C = \hat C(\rho, \tilde\rho) > 0$ such that

$$\begin{aligned} &|I_{\tilde\rho}(r) - I_\rho(r)| \\ &\le c_{10}\, \hat C \Big(\alpha^{\frac{\sigma(1+2\sigma)}{3+2\sigma}} (\eta + \alpha^{-\frac{2\sigma}{3+2\sigma}} \eta^{\frac{5+2\sigma}{3+2\sigma}}) + \eta^{\frac{4(1+\sigma)}{3+2\sigma}} + \alpha^\sigma\, \eta^{\frac{1+2\sigma}{3+2\sigma}} \Big) R\, |\tilde\rho(r) - \rho(r)| \\ &\quad + c_{10}\, \hat C \Big(R^2 (\eta^2 + \alpha^{\frac{2\sigma}{3+2\sigma}}\, \eta^{\frac{4(1+\sigma)}{3+2\sigma}}) + \alpha^{\frac{2\sigma}{3+2\sigma}}\, \eta^{\frac{1+2\sigma}{3+2\sigma}} \Big) \int_0^r |\tilde\rho(s) - \rho(s)|\, ds. \end{aligned} \tag{4.3}$$

Note that c_{10} is an absolute constant, while $\hat C$ depends on ρ and $\tilde\rho$.
(e) There is a constant $\bar C = \bar C(\alpha, \eta) > 0$ with the following property. Let $R > 0$ and $\rho, \tilde\rho \in C([0,R])$ be such that $\frac{\eta}{2} \le \rho(r) \le \eta$, $\frac{\eta}{2} \le \tilde\rho(r) \le \eta$ and $\frac{2m(r)}{r} \le \frac{1}{2}$, $\frac{2\tilde m(r)}{r} \le \frac{1}{2}$ for $r \in [0,R]$. Then

$$|I_{\tilde\rho}(r) - I_\rho(r)| \le \bar C R\, |\tilde\rho(r) - \rho(r)| + \bar C (R^2 + 1) \int_0^r |\tilde\rho(s) - \rho(s)|\, ds. \tag{4.4}$$

Proof: (a) Since $G''(\varepsilon) > 0$ for $\varepsilon \in]0,\alpha[$ by Lemma 7.5 and $-\zeta(r) = -\mathcal{G}'(4\pi\kappa\rho(r)) \in]0,\alpha[$ according to Lemma 7.8, we deduce that $G''(-\zeta(r)) > 0$. Also $G(\varepsilon) > 0$ for $\varepsilon \in]0,\alpha[$ by Lemma 7.3, so that $G(-\zeta(r)) > 0$. As a consequence, we get $I_\rho(r) > 0$ for $r \in]0,R]$. (b) This follows from (a). (c) From $0 \le \frac{2m(r)}{r} \le \frac{1}{2}$ we obtain $\frac{1}{1-\frac{2m(r)}{r}} \in [1,2]$ for $r \in [0,R]$. Moreover, $0 \le m(r) = 4\pi \int_0^r s^2 \rho(s)\,ds \le \frac{4\pi}{3}\,\eta\, r^3$, so that $\frac{m(r)}{r^2} \le \frac{4\pi}{3}\,\eta\, r \le \frac{4\pi}{3}\,\eta\, R$ for $r \in [0,R]$. Thus we deduce from (3.19) and (3.17) that

$$\begin{aligned} 0 &\le \kappa\, G''(-\zeta(r))\, \frac{m(r)}{r^2}\, \frac{1}{1-\frac{2m(r)}{r}}\,(-\zeta(r)) \\ &\le \kappa\, c_8\, \alpha^{-\frac{3}{3+2\sigma}} \rho(r)^{\frac{1+2\sigma}{3+2\sigma}}\, \frac{8\pi}{3}\, \eta\, R\, \alpha \\ &\le \frac{8\pi}{3}\, \kappa\, c_8\, \alpha^{\frac{2\sigma}{3+2\sigma}} \eta^{\frac{4(1+\sigma)}{3+2\sigma}}\, R. \end{aligned}$$

Similarly, by (3.19) and (3.18),

$$\begin{aligned} 0 &\le G''(-\zeta(s))\, \frac{s}{1-\frac{2m(s)}{s}}\, G(-\zeta(s)) \\ &\le c_8\, \alpha^{-\frac{3}{3+2\sigma}} \rho(r)^{\frac{1+2\sigma}{3+2\sigma}}\, 2R\, c_8\, \alpha^{\frac{3}{3+2\sigma}} \rho(r)^{\frac{5+2\sigma}{3+2\sigma}} \\ &\le 2c_8^2\, \eta^2\, R, \end{aligned}$$

so that altogether

$$I_\rho(r) \le \Big(\frac{8\pi}{3}\, \kappa\, c_8\, \alpha^{\frac{2\eta}{3+2\sigma}} \eta^{\frac{4(1+\sigma)}{3+2\sigma}} + 2c_8^2\, \eta^2\Big)\, R,$$

which shows that we can take $c_9 = \max\{\frac{8\pi}{3}\,\kappa\, c_8, 2c_8^2\}$. (d) Let $\delta > 0$ be such that $\delta \le \rho(r) \le \eta$, $\delta \le \tilde{\rho}(r) \le \eta$ and $\frac{2m(r)}{r} \le 1-\delta$, $\frac{2\tilde{m}(r)}{r} \le 1-\delta$ for $r \in [0,R]$. Then there is $q \in [\frac{1}{\sqrt{2}}, 1[$ (that can be calculated from δ) so that $-\zeta(r) = -\mathcal{G}'(4\pi\kappa\rho(r)) \in [\frac{1}{\sqrt{2}}\alpha, q\alpha]$ as well as $-\tilde{\zeta}(r) = -\mathcal{G}'(4\pi\kappa\tilde{\rho}(r)) \in [\frac{1}{\sqrt{2}}\alpha, q\alpha]$; for this, also recall (3.17). Given this q, let $C_q > 0$ denote the associated constant from Corollary 7.6. We have

$$\begin{aligned} &|I_{\tilde{\rho}}(r) - I_\rho(r)| \\ &\quad\le |G''(-\tilde{\zeta}(r)) - G''(-\zeta(r))| \Big[\kappa\, \frac{\tilde{m}(r)}{r^2}\, \frac{1}{1-\frac{2\tilde{m}(r)}{r}}\,(-\tilde{\zeta}(r)) + \frac{r}{1-\frac{2\tilde{m}(r)}{r}}\, G(-\tilde{\zeta}(r))\Big] \\ &\qquad + \kappa\, |G''(-\zeta(r))| \Big|\frac{\tilde{m}(r)}{r^2}\, \frac{1}{1-\frac{2\tilde{m}(r)}{r}}\,(-\tilde{\zeta}(r)) - \frac{m(r)}{r^2}\, \frac{1}{1-\frac{2m(r)}{r}}\,(-\zeta(r))\Big| \\ &\qquad + |G''(-\zeta(r))| \Big|\frac{r}{1-\frac{2\tilde{m}(r)}{r}}\, G(-\tilde{\zeta}(r)) - \frac{r}{1-\frac{2m(r)}{r}}\, G(-\zeta(r))\Big| \\ &\quad\le |G''(-\tilde{\zeta}(r)) - G''(-\zeta(r))| \Big[\kappa\, \frac{\tilde{m}(r)}{r^2}\, \frac{1}{1-\frac{2\tilde{m}(r)}{r}}\,(-\tilde{\zeta}(r)) + \frac{r}{1-\frac{2\tilde{m}(r)}{r}}\, G(-\tilde{\zeta}(r))\Big] \\ &\qquad + \kappa\, |G''(-\zeta(r))|\, \frac{\tilde{m}(r)}{r^2}\, \frac{1}{1-\frac{2\tilde{m}(r)}{r}}\, |\tilde{\zeta}(r) - \zeta(r)| \end{aligned}$$

$$
\begin{aligned}
&+ 2\kappa\, |G''(-\zeta(r))|\, \frac{\tilde m(r)}{r^3}\, (-\zeta(r))\, \frac{|\tilde m(r) - m(r)|}{(1-\frac{2\tilde m(r)}{r})(1-\frac{2m(r)}{r})} \\
&+ \kappa\, |G''(-\zeta(r))|\, (-\zeta(r))\, \frac{1}{1-\frac{2m(r)}{r}}\, \frac{|\tilde m(r) - m(r)|}{r^2} \\
&+ |G''(-\zeta(r))|\, \frac{r}{1-\frac{2\tilde m(r)}{r}}\, |G(-\tilde\zeta(r)) - G(-\zeta(r))| \\
&+ 2\, |G''(-\zeta(r))|\, |G(-\zeta(r))| \frac{|\tilde m(r) - m(r)|}{(1-\frac{2\tilde m(r)}{r})(1-\frac{2m(r)}{r})} \\
=:\ & T_1 + T_2 + T_3 + T_4 + T_5 + T_6.
\end{aligned}
$$

Henceforth $C > 0$ will denote a generic constant (that is independent of R, α, η, δ). To bound T_1, by (3.14) and (3.15) we have

$$
\begin{aligned}
|\tilde\zeta(r) - \zeta(r)| &= |\mathcal{G}'(4\pi\kappa\tilde\rho(r)) - \mathcal{G}'(4\pi\kappa\rho(r))| \\
&\le 4\pi\kappa\, (\max_{s\in[4\pi\kappa\delta, 4\pi\kappa\eta]} \mathcal{G}''(s))\, |\tilde\rho(r) - \rho(r)| \\
&\le 4\pi\kappa\, c_4\, \alpha^{\frac{3}{3+2\sigma}}\, (4\pi\kappa\delta)^{-\frac{1+2\sigma}{3+2\sigma}}\, |\tilde\rho(r) - \rho(r)| \\
&\le C\, \alpha^{\frac{3}{3+2\sigma}}\, \delta^{-\frac{1+2\sigma}{3+2\sigma}}\, |\tilde\rho(r) - \rho(r)|. \qquad (4.5)
\end{aligned}
$$

Therefore, using Corollary 7.6,

$$
\begin{aligned}
|G''(-\tilde\zeta(r)) - G''(-\zeta(r))| &\le C_q\, \alpha^{-(2-\sigma)}\, |\tilde\zeta(r) - \zeta(r)| \\
&\le CC_q\, \alpha^{-\frac{(\sigma+1)(3-2\sigma)}{3+2\sigma}}\, \delta^{-\frac{1+2\sigma}{3+2\sigma}}\, |\tilde\rho(r) - \rho(r)|.
\end{aligned}
$$

Moreover,

$$
\tilde m(r) = 4\pi \int_0^r s^2 \rho(s)\, ds \le \frac{4\pi}{3}\, \eta\, r^3 \qquad (4.6)
$$

and

$$
G(-\tilde\zeta(r)) \le c_8\, \alpha^{\frac{3}{3+2\sigma}}\, \tilde\rho(r)^{\frac{5+2\sigma}{3+2\sigma}} \le c_8\, \alpha^{\frac{3}{3+2\sigma}}\, \eta^{\frac{5+2\sigma}{3+2\sigma}} \qquad (4.7)
$$

by (3.18). Taking these estimates together, if follows that

$$
T_1 \le CC_q\, \alpha^{\frac{\sigma(1+2\sigma)}{3+2\sigma}}\, \delta^{-\frac{4(1+\sigma)}{3+2\sigma}}\, (\eta + \alpha^{-\frac{2\sigma}{3+2\sigma}} \eta^{\frac{5+2\sigma}{3+2\sigma}})\, R\, |\tilde\rho(r) - \rho(r)|.
$$

For T_2, we use (3.19) to obtain

$$
|G''(-\zeta(r))| \le c_8\, \alpha^{-\frac{3}{3+2\sigma}}\, \tilde\rho(r)^{\frac{1+2\sigma}{3+2\sigma}} \le c_8\, \alpha^{-\frac{3}{3+2\sigma}}\, \eta^{\frac{1+2\sigma}{3+2\sigma}}. \qquad (4.8)
$$

Therefore (4.6) and (4.5) yield

$$
T_2 \le C\, \delta^{-\frac{4(1+\sigma)}{3+2\sigma}}\, \eta^{\frac{4(1+\sigma)}{3+2\sigma}}\, R\, |\tilde\rho(r) - \rho(r)|.
$$

Next,

$$|\tilde{m}(r)-m(r)| \le 4\pi \int_0^r s^2\,|\tilde\rho(s)-\rho(s)|\,ds \le 4\pi r^2 \int_0^r |\tilde\rho(s)-\rho(s)|\,ds, \tag{4.9}$$

and thus from (4.8) and (4.6) we get

$$T_3 \le C\,\alpha^{\frac{2\sigma}{3+2\sigma}}\,\delta^{-2}\,\eta^{\frac{4(1+\sigma)}{3+2\sigma}}\,R^2 \int_0^r |\tilde\rho(s)-\rho(s)|\,ds.$$

For T_4 we can argue in a similar way, and due to $r^{-2}|\tilde{m(r)}-m(r)| \le 4\pi\int_0^r |\tilde\rho(s)-\rho(s)|\,ds$ we also have

$$T_4 \le C\,\alpha^{\frac{2\sigma}{3+2\sigma}}\,\delta^{-1}\,\eta^{\frac{1+2\sigma}{3+2\sigma}} \int_0^r |\tilde\rho(s)-\rho(s)|\,ds.$$

Regarding T_5, by using (3.12) and (4.5) we deduce

$$\begin{aligned}
|G(-\tilde\zeta(r))-G(-\zeta(r))| &\le \Big(\max_{\varepsilon\in[\frac{1}{\sqrt2}\alpha,q\alpha]}|G'(\varepsilon)|\Big)\,|\tilde\zeta(r)-\zeta(r)| \\
&\le C\alpha^{-3/2}\Big(\max_{\varepsilon\in[\frac{1}{\sqrt2}\alpha,q\alpha]}(\alpha-\varepsilon)^{3/2+\sigma}\Big)\,\alpha^{\frac{3}{3+2\sigma}}\,\delta^{-\frac{1+2\sigma}{3+2\sigma}}\,|\tilde\rho(r)-\rho(r)| \\
&\le C\alpha^{\frac{2\sigma^2+3\sigma+3}{3+2\sigma}}\,\delta^{-\frac{1+2\sigma}{3+2\sigma}}\,|\tilde\rho(r)-\rho(r)|.
\end{aligned}$$

Hence, as a consequence of (4.8),

$$T_5 \le C\alpha^{\sigma}\,\delta^{-\frac{4(1+\sigma)}{3+2\sigma}}\,\eta^{\frac{1+2\sigma}{3+2\sigma}}\,R\,|\tilde\rho(r)-\rho(r)|.$$

Lastly, the bound

$$T_6 \le C\,\delta^{-2}\eta^2\,R^2 \int_0^r |\tilde\rho(s)-\rho(s)|\,ds$$

is again due to (4.8), (4.7) and (4.9). Taking together all the above estimates on $T_1,\dots,T_6$, we arrive at

$$\begin{aligned}
&|I_{\tilde\rho}(r)-I_\rho(r)| \\
&\quad\le C\delta^{-\frac{4(1+\sigma)}{3+2\sigma}}\Big(C_q\,\alpha^{\frac{\sigma(1+2\sigma)}{3+2\sigma}}\,(\eta+\alpha^{-\frac{2\sigma}{3+2\sigma}}\eta^{\frac{5+2\sigma}{3+2\sigma}})+\eta^{\frac{4(1+\sigma)}{3+2\sigma}}+\alpha^{\sigma}\,\eta^{\frac{1+2\sigma}{3+2\sigma}}\Big)\,R\,|\tilde\rho(r)-\rho(r)| \\
&\qquad+C\Big(\delta^{-2}R^2(\eta^2+\alpha^{\frac{2\sigma}{3+2\sigma}}\,\eta^{\frac{4(1+\sigma)}{3+2\sigma}})+\alpha^{\frac{2\sigma}{3+2\sigma}}\,\delta^{-1}\,\eta^{\frac{1+2\sigma}{3+2\sigma}}\Big)\int_0^r |\tilde\rho(s)-\rho(s)|\,ds,
\end{aligned} \tag{4.10}$$

after some regrouping of terms, which yields (4.3). (e) This follows from an inspection of the argument for (d) and from (4.10). We can take $\delta=\min\{\frac{\eta}{2},\frac12\}$ everywhere, and since $-\zeta(r)=-\mathcal{G}'(4\pi\kappa\rho(r))\in[\frac{1}{\sqrt2}\alpha,-\mathcal{G}'(2\pi\kappa\eta)]$, we need to take $q=-\alpha^{-1}\,\mathcal{G}'(2\pi\kappa\eta)$ when we apply Corollary 7.6 to determine C_q. Note that with some efforts the constant $\bar C=\bar C(\alpha,\eta)$ could be calculated explicitly, since C_q is explicit. □

First we have to deal with uniqueness of solutions to (4.1).

Lemma 4.2 *Let $R > 0$ and let $\rho, \tilde{\rho} \in C([0, R])$ be solutions of (4.1) such that $0 < \rho(r) \leq \eta$, $0 < \tilde{\rho}(r) \leq \eta$ and $\frac{2m(r)}{r} < 1$, $\frac{2\tilde{m}(r)}{r} < 1$ for $r \in [0, R]$. Then $\rho(r) = \tilde{\rho}(r)$ for $r \in [0, R]$.*

Proof: Due to (4.3) in Lemma 4.1(d) there is a constant $\tilde{C} = \tilde{C}(\rho, \tilde{\rho}, R, \alpha, \eta) > 0$ such that

$$|I_{\tilde{\rho}}(r) - I_\rho(r)| \leq \tilde{C}\Big(|\tilde{\rho}(r) - \rho(r)| + \int_0^r |\tilde{\rho}(s) - \rho(s)|\,ds\Big), \quad r \in [0, R],$$

owing to (4.3). By (4.1) this yields

$$\begin{aligned} |\tilde{\rho}(r) - \rho(r)| &\leq \frac{1}{4\pi\kappa^2}\int_0^r |I_{\tilde{\rho}}(s) - I_\rho(s)|\,ds \\ &\leq \frac{\tilde{C}}{4\pi\kappa^2}\int_0^r \Big(|\tilde{\rho}(s) - \rho(s)| + \int_0^s |\tilde{\rho}(\tau) - \rho(\tau)|\,d\tau\Big)\,ds \end{aligned}$$

for $r \in [0, R]$. Denote $\Delta(r) = \max_{s\in[0,r]} |\tilde{\rho}(s) - \rho(s)|$. Since Δ is monotone, it follows that

$$|\tilde{\rho}(r) - \rho(r)| \leq \frac{\tilde{C}}{4\pi\kappa^2}\int_0^r (\Delta(s) + s\Delta(s))\,ds \leq \frac{\tilde{C}}{4\pi\kappa^2}\,(1 + R)\int_0^r \Delta(s)\,ds$$

for $r \in [0, R]$. Fix $r \in [0, R]$ and $\bar{r} \in [0, r]$. Then

$$|\tilde{\rho}(\bar{r}) - \rho(\bar{r})| \leq \frac{\tilde{C}}{4\pi\kappa^2}\,(1 + R)\int_0^{\bar{r}} \Delta(s)\,ds \leq \frac{\tilde{C}}{4\pi\kappa^2}\,(1 + R)\int_0^r \Delta(s)\,ds$$

yields

$$\Delta(r) \leq \frac{\tilde{C}}{4\pi\kappa^2}\,(1 + R)\int_0^r \Delta(s)\,ds,$$

and hence Gronwall's inequality applies. □

Next comes the local existence of a solution to (4.1).

Lemma 4.3 *Let $c_9 > 0$ be the constant from Lemma 4.1(c) and denote by $\bar{C} = \bar{C}(\alpha, \eta) > 0$ the constant from Lemma 4.1(e). Suppose that $R \in]0, 1]$ is so small that*

$$(\alpha^{\frac{2\sigma}{3+2\sigma}}\eta^{\frac{1+2\sigma}{3+2\sigma}} + \eta)\,R^2 \leq \frac{2\pi\kappa^2}{c_9}, \quad \eta\,R^2 \leq \frac{3}{16\pi} \quad \textit{and} \quad \bar{C}R^2 \leq \frac{2\pi\kappa^2}{3}. \tag{4.11}$$

Then (4.1) has a (unique) solution $\rho \in C([0, R])$ so that $\rho(0) = \eta$, $\frac{\eta}{2} \leq \rho(r) \leq \eta$ and $\frac{2m(r)}{r} \leq \frac{1}{2}$ for $r \in [0, R]$.

Proof: This is a standard application of Banach's fixed point theorem on the closed set

$$D = \Big\{\rho \in C([0,R]) : \rho(0) = \eta, \frac{\eta}{2} \le \rho(r) \le \eta \text{ and } \frac{2m(r)}{r} \le \frac{1}{2}, r \in [0,R]\Big\}$$

in the Banach space $X = C([0,R])$, and for the operator

$$F : D \to D, \quad (F\rho)(r) = \eta - \frac{1}{4\pi\kappa^2}\int_0^r I_\rho(s)\,ds,$$

with I_ρ given by (4.2). First we will show that $F : D \to D$ is well-defined. For, let $\rho \in D$. Clearly $(F\rho)(0) = 0$, and since $I_\rho(s) > 0$ for $s \in]0,R]$ by Lemma 4.1(a), we also have $(F\rho)(r) \le \eta$ for $r \in [0,R]$. What concerns the lower bound, from Lemma 4.1(c) we recall that

$$I_\rho(r) \le c_9(\alpha^{\frac{2\sigma}{3+2\sigma}}\eta^{\frac{4(1+\sigma)}{3+2\sigma}} + \eta^2)\,R$$

for $r \in [0,R]$. Hence

$$(F\rho)(r) \ge \eta - \frac{1}{4\pi\kappa^2}\,c_9(\alpha^{\frac{2\sigma}{3+2\sigma}}\eta^{\frac{4(1+\sigma)}{3+2\sigma}} + \eta^2)\,R^2 \ge \frac{\eta}{2}$$

for $r \in [0,R]$, where we have used (4.11). Next, for the mass generated by $F\rho$ we obtain

$$\frac{2}{r}\,4\pi\int_0^r s^2(F\rho)(s)\,ds \le \frac{8\pi}{3r}\,\eta\,r^3 \le \frac{8\pi}{3}\,\eta\,R^2 \le \frac{1}{2}$$

for $r \in [0,R]$, once again by (4.11). Therefore we have seen that indeed $F(D) \subset D$ is verified. For the contraction property, if $\rho, \tilde\rho \in D$ and $r \in [0,R]$, then

$$|(F\rho)(r) - (F\tilde\rho)(r)| \le \frac{1}{4\pi\kappa^2}\int_0^r |I_{\tilde\rho}(s) - I_\rho(s)|\,ds.$$

Due to (4.4) from Lemma 4.1(e) we have

$$\begin{aligned} |I_{\tilde\rho}(s) - I_\rho(s)| &\le \bar C R\,|\tilde\rho(s) - \rho(s)| + \bar C(R^2+1)\int_0^s |\tilde\rho(\tau) - \rho(\tau)|\,d\tau \\ &\le \bar C R\,(R^2+2)\,\|\tilde\rho - \rho\|_X, \end{aligned}$$

so that

$$\|F\rho - F\tilde\rho\|_X \le \frac{1}{4\pi\kappa^2}\,\bar C R^2\,(R^2+2)\,\|\tilde\rho - \rho\|_X \le \frac{1}{2}\,\|\tilde\rho - \rho\|_X,$$

once again using (4.11). Thus F is a contraction. $\Box$

Remark 4.4 If $r_0 > 0$ and if $\eta_0 > 0$ is sufficiently small, then it can be shown in an analogous way as in Lemma 4.3, that for small $R > 0$ a (unique) solution $\rho \in C([r_0, r_0+R])$ of (4.1) replaced by $\rho(r) = \eta_0 - \frac{1}{4\pi\kappa^2}\int_{r_0}^r [\ldots]\,ds$ does exists so that $\rho(r_0) = \eta_0$, $\frac{\eta_0}{2} \le \rho(r) \le \eta_0$ and $\frac{2m(r)}{r} \le \frac{1}{2}$ for $r \in [r_0, r_0+R]$. $\diamondsuit$

We are now in a position to prove Theorem 3.3.

Proof of Theorem 3.3: Consider the maximal solution ρ_* of (4.1) such that $0 < \rho_*(r) \le \eta$ and $\frac{2m_*(r)}{r} < 1$ holds, which is defined on some interval $[0, R_*[$, where $0 < R_* \le \infty$. Such a maximal solution does exist (by Zorn's lemma), since there is some solution with these properties on some interval $[0, R]$ by Lemma 4.3, and solutions with these properties are unique by Lemma 4.2. Furthermore, Remark 4.4 excludes the case that the maximal interval of existence could be closed, i.e., equal to some $[0, R_*]$. According to Lemma 4.1(b) we also know that $\rho_* \in C^1([0, R_*[)$ satisfies $\rho_*'(r) < 0$ for $r \in [0, R_*[$. Let $\zeta_*(r) = \mathcal{G}'(4\pi\kappa\rho_*(r))$ be as before. ¿From the choice of η and (3.17) in Lemma 3.2 we infer that $-\zeta_*(r) \in [\frac{1}{\sqrt{2}}\alpha, \alpha]$ for $r \in [0, R_*[$. Differentiating (4.1), we obtain

$$\rho_*'(r) = -\frac{1}{4\pi\kappa^2}\, G''(-\zeta_*(r))\,\Big[\kappa\,\frac{m_*(r)}{r^2}\,\frac{1}{1-\frac{2m_*(r)}{r}}\,(-\zeta_*(r)) + \frac{r}{1-\frac{2m_*(r)}{r}}\,G(-\zeta_*(r))\Big], \quad r \in [0, R_*[. \tag{4.12}$$

As a consequence,

$$\begin{aligned} \kappa\,\zeta_*'(r) &= 4\pi\kappa^2\,\mathcal{G}''(4\pi\kappa\rho_*(r))\,\rho_*'(r) \\ &= \mathcal{G}''(4\pi\kappa\rho_*(r))\;G''(-\zeta_*(r))\,\Big[\kappa\,\frac{m_*(r)}{r^2}\,\frac{1}{1-\frac{2m_*(r)}{r}}\,\zeta_*(r) - \frac{r}{1-\frac{2m_*(r)}{r}}\,G(-\zeta_*(r))\Big]. \end{aligned} \tag{4.13}$$

For $s = 4\pi\kappa\rho_*(r)$ and $u = -s = -4\pi\kappa\rho_*(r)$ we have $\varepsilon(u) = \hat{G}'(u)$ by (7.14). Hence $\mathcal{G}(s) = \hat{G}(-s)$ shows that $\varepsilon(-s) = \varepsilon(u) = \hat{G}'(u) = -\mathcal{G}'(-u) = -\mathcal{G}'(s) = -\zeta_*(r)$. Therefore we can apply (7.16), and (4.13) simplifies to

$$\zeta_*'(r) = \frac{m_*(r)}{r^2}\,\frac{1}{1-\frac{2m_*(r)}{r}}\,\zeta_*(r) - \kappa^{-1}\frac{r}{1-\frac{2m_*(r)}{r}}\,G(-\zeta_*(r)), \quad r \in [0, R_*[. \tag{4.14}$$

First we are going to show that $R_* < \infty$ is verified, and for this we will use an argument similar to the one in [28, Prop. 4.3]. Suppose that we have $R_* = \infty$. Then (4.1) holds for $r \in [0, \infty[$, and moreover $0 < \rho_*(r) \le \eta$ and $\frac{2m_*(r)}{r} < 1$ for $r \in [0, \infty[$. We apply the change of variables $s = r^3$, $r = s^{1/3}$, and define

$$\hat{\rho}(s) = \rho_*(r) = \rho_*(s^{1/3}), \quad \hat{m}(s) = m_*(r) = m_*(s^{1/3}), \quad \hat{\zeta}(s) = \zeta_*(r) = \zeta_*(s^{1/3}).$$

Note that always $2\hat{m}(s) < s^{1/3}$ due to $\frac{2m_*(r)}{r} < 1$. In addition,

$$\hat{m}'(s) = \frac{1}{3}\,s^{-2/3}\,m_*'(s^{1/3}) = \frac{1}{3}\,s^{-2/3}\,4\pi s^{2/3}\,\rho_*(s^{1/3}) = \frac{4\pi}{3}\,\rho_*(s^{1/3}),$$

and thus

$$\hat{m}''(s) = \frac{4\pi}{9}\,s^{-2/3}\,\rho_*'(s^{1/3}) < 0$$

for $s \in]0,\infty[$, which implies that $\hat{m}$ is concave on $[0,\infty[$. Hence $(s-s_0)\hat{m}'(s) \le \hat{m}(s) - \hat{m}(s_0)$ for $0 \le s_0 \le s$, so that $\hat{m}(0) = 0$ yields

$$\hat{\rho}(s) = \frac{3}{4\pi}\,\hat{m}'(s) \le \frac{3}{4\pi}\,\frac{\hat{m}(s)}{s}, \quad s \in]0,\infty[. \tag{4.15}$$

Moreover, in the variable s, (4.14) is found to read as

$$\hat{\zeta}'(s) = \frac{\hat{m}(s)}{3s}\,\frac{1}{(s^{1/3} - 2\hat{m}(s))}\,\hat{\zeta}(s) - \frac{1}{3\kappa}\,\frac{1}{(s^{1/3} - 2\hat{m}(s))}\,G(-\hat{\zeta}(s)), \quad s \in]0,\infty[. \tag{4.16}$$

Since $\hat{\zeta}(s) < 0$ and $\hat{m}(s) \ge 0$ we have

$$\frac{\hat{m}(s)}{3s}\,\frac{1}{(s^{1/3} - 2\hat{m}(s))}\,\hat{\zeta}(s) \le \frac{\hat{m}(s)}{3s^{4/3}}\,\hat{\zeta}(s)$$

and hence $G(-\hat{\zeta}(s)) \ge 0$ in conjunction with (4.16) leads to

$$\hat{\zeta}'(s) \le \frac{\hat{m}(s)}{3s^{4/3}}\,\hat{\zeta}(s), \quad s \in]0,\infty[. \tag{4.17}$$

Similarly, from

$$-\frac{1}{3\kappa}\,\frac{1}{(s^{1/3} - 2\hat{m}(s))}\,G(-\hat{\zeta}(s)) \le -\frac{1}{3\kappa}\,\frac{1}{s^{1/3}}\,G(-\hat{\zeta}(s))$$

together with (4.16) we find that

$$\hat{\zeta}'(s) \le -\frac{1}{3\kappa}\,\frac{1}{s^{1/3}}\,G(-\hat{\zeta}(s)), \quad s \in]0,\infty[, \tag{4.18}$$

holds. Those two differential inequalities (4.17) and (4.18) for $\hat{\zeta}$ can be used as follows. Due to (3.11) and $-\hat{\zeta}(s) \in [\frac{1}{\sqrt{2}}\alpha, \alpha]$ we have

$$G(-\hat{\zeta}(s)) \ge c_1\,\alpha^{-3/2}\,(\alpha + \hat{\zeta}(s))^{5/2+\sigma}.$$

Therefore (4.18) yields

$$\begin{aligned}
\frac{d}{ds}\,(\alpha + \hat{\zeta}(s))^{-(3/2+\sigma)} &= -\Big(\frac{3+2\sigma}{2}\Big)\,(\alpha + \hat{\zeta}(s))^{-(5/2+\sigma)}\,\hat{\zeta}'(s) \\
&\ge \Big(\frac{3+2\sigma}{6\kappa}\Big)\,\frac{1}{s^{1/3}}\,(\alpha + \hat{\zeta}(s))^{-(5/2+\sigma)}\,G(-\hat{\zeta}(s)) \\
&\ge c_1\Big(\frac{3+2\sigma}{6\kappa}\Big)\,\alpha^{-3/2}\,\frac{1}{s^{1/3}}, \quad s \in]0,\infty[.
\end{aligned} \tag{4.19}$$

Now we need to distinguish two cases. (i) If $\hat{m}(s) \le 1$ for all $s \in [0,\infty[$, then $\hat{M} = \lim_{s\to\infty} \hat{m}(s) \in]0,1]$ does exist, since $\hat{m}$ is increasing. Here we take $s_* > 0$ so large that

$$\hat{m}(s) \ge \frac{\hat{M}}{2} \quad \text{and} \quad s^{\frac{3-2\sigma}{6}} > \frac{3 \cdot 20^{\frac{3+2\sigma}{2}}}{4\pi}\,c_6^{\frac{3+2\sigma}{2}}\,\alpha^{-\sigma}\,\Big(\frac{2}{\hat{M}}\Big)^{\frac{3+2\sigma}{2}} \tag{4.20}$$

holds for all $s \geq s_*$; recall that $3 - 2\sigma > 0$. (ii) If $\hat{m}(s) > 1$ for $s \geq s_0$ (with an appropriate $s_0 > 0$) is verified, then we determine $s_* \geq s_0$ such that

$$s^{\frac{3-2\sigma}{6}} > \frac{3 \cdot 20^{\frac{3+2\sigma}{2}}}{4\pi} \, c_6^{\frac{3+2\sigma}{2}} \, \alpha^{-\sigma} \tag{4.21}$$

for $s \geq s_*$. Consider $s \geq s_*$. Integrating (4.19), we arrive at

$$(\alpha + \hat{\zeta}(s))^{-(3/2+\sigma)} - (\alpha + \hat{\zeta}(s_*))^{-(3/2+\sigma)} \geq c_{11} \, \alpha^{-3/2} \, (s^{2/3} - s_*^{2/3}), \quad s \in [s_*, \infty[,$$

where $c_{11} = c_1(\frac{3+2\sigma}{4\kappa})$. This can be recast as

$$0 \leq \alpha + \hat{\zeta}(s) \leq (\alpha + \hat{\zeta}(s_*)) \Big[1 + c_{11} \, \alpha^{-3/2} \, (\alpha + \hat{\zeta}(s_*))^{3/2+\sigma} \, (s^{2/3} - s_*^{2/3}) \Big]^{-\frac{2}{3+2\sigma}}, \quad s \in [s_*, \infty[, \tag{4.22}$$

Next, $-\hat{\zeta}(s) \geq \frac{1}{\sqrt{2}} \, \alpha \geq \frac{1}{2} \, \alpha$. Thus, owing to (4.17),

$$\hat{\zeta}'(s) \leq -\frac{\alpha}{6} \, \frac{\hat{m}(s)}{s^{4/3}}, \quad s \in]0, \infty[.$$

Hence for $s \geq s_*$ and $\Delta > 0$ the monotonicity of $\hat{m}$ implies that

$$\hat{\zeta}(s + \Delta) - \hat{\zeta}(s) \leq -\frac{\alpha}{6} \int_s^{\Delta+s} \frac{\hat{m}(\tau)}{\tau^{4/3}} \, d\tau \leq -\frac{\alpha}{2} \, \hat{m}(s) \, (s^{-1/3} - (\Delta + s)^{-1/3}). \tag{4.23}$$

Applying (4.23) to $s = 2s_*$ and $\Delta = s_*$, we find

$$\alpha + \hat{\zeta}(3s_*) \leq \alpha + \hat{\zeta}(2s_*) - \frac{\alpha}{2} \, \frac{\hat{m}(2s_*)}{s_*^{1/3}} \left(\frac{1}{2^{1/3}} - \frac{1}{3^{1/3}} \right) \leq \alpha + \hat{\zeta}(2s_*) - \frac{\alpha}{20} \, \frac{\hat{m}(2s_*)}{s_*^{1/3}}.$$

Thus taking $s = 2s_*$ in (4.22) to bound $\alpha + \hat{\zeta}(2s_*)$ on the right-hand side, we get

$$\begin{aligned} \alpha + \hat{\zeta}(3s_*) & \leq (\alpha + \hat{\zeta}(s_*)) \Big[1 + c_{11} \, \alpha^{-3/2} \, (\alpha + \hat{\zeta}(s_*))^{3/2+\sigma} \, s_*^{2/3} (2^{2/3} - 1) \Big]^{-\frac{2}{3+2\sigma}} - \frac{\alpha}{20} \, \frac{\hat{m}(2s_*)}{s_*^{1/3}} \\ & \leq (\alpha + \hat{\zeta}(s_*)) \Big[1 + \frac{c_{11}}{2} \, \alpha^{-3/2} \, (\alpha + \hat{\zeta}(s_*))^{3/2+\sigma} \, s_*^{2/3} \Big]^{-\frac{2}{3+2\sigma}} - \frac{\alpha}{20} \, \frac{\hat{m}(2s_*)}{s_*^{1/3}}. \end{aligned} \tag{4.24}$$

¿From (3.16) and $\rho_*(r) > 0$ for $r \in [0, \infty[$ we deduce that actually

$$\alpha + \hat{\zeta}(3s_*) = \alpha + \zeta_*((3s_*)^{1/3}) \geq c_5 \, \alpha^{\frac{3}{3+2\sigma}} \rho_*((3s_*)^{1/3})^{\frac{2}{3+2\sigma}} > 0$$

for the left-hand side of (4.24). However, we are going to show that

$$s_*^{\frac{3+2\sigma}{6}} (\alpha + \hat{\zeta}(s_*))^{\frac{3+2\sigma}{2}} < \left(\frac{\alpha}{20} \right)^{\frac{3+2\sigma}{2}} \Big[1 + \frac{c_{11}}{2} \, \alpha^{-3/2} \, (\alpha + \hat{\zeta}(s_*))^{3/2+\sigma} \, s_*^{2/3} \Big] \, \hat{m}(2s_*)^{\frac{3+2\sigma}{2}}, \tag{4.25}$$

and then the desired contradiction will be reached. To establish (4.25), we use (3.16) and (4.15) to estimate

$$\begin{aligned} s_*^{\frac{3+2\sigma}{6}}(\alpha+\hat{\zeta}(s_*))^{\frac{3+2\sigma}{2}} &= s_*^{\frac{3+2\sigma}{6}}(\alpha+\zeta_*(s_*^{1/3}))^{\frac{3+2\sigma}{2}} \le c_6^{\frac{3+2\sigma}{2}}\alpha^{3/2}\, s_*^{\frac{3+2\sigma}{6}}\rho_*(s_*^{1/3}) = c_6^{\frac{3+2\sigma}{2}}\alpha^{3/2}\, s_*^{\frac{3+2\sigma}{6}}\hat{\rho}(s_*) \\ &\le \frac{3}{4\pi}\, c_6^{\frac{3+2\sigma}{2}}\alpha^{3/2}\, s_*^{\frac{3+2\sigma}{6}}\frac{\hat{m}(s_*)}{s_*} = \frac{3}{4\pi}\, c_6^{\frac{3+2\sigma}{2}}\alpha^{3/2}\, s_*^{-\frac{3-2\sigma}{6}}\hat{m}(s_*). \end{aligned} \tag{4.26}$$

At this point we have to come back to the cases (i) and (ii) that we distinguished above, and which led to different choices of s_*. Case (i): $\hat{m}(s) \le 1$ for all $s \in [0,\infty[$. Then by (4.26) and (4.20),

$$\begin{aligned} s_*^{\frac{3+2\sigma}{6}}(\alpha+\hat{\zeta}(s_*))^{\frac{3+2\sigma}{2}} &\le \frac{3}{4\pi}\, c_6^{\frac{3+2\sigma}{2}}\alpha^{3/2}\, s_*^{-\frac{3-2\sigma}{6}} \\ &< \left(\frac{\alpha}{20}\right)^{\frac{3+2\sigma}{2}}\left(\frac{\hat{M}}{2}\right)^{\frac{3+2\sigma}{2}} \le \left(\frac{\alpha}{20}\right)^{\frac{3+2\sigma}{2}}\hat{m}(2s_*)^{\frac{3+2\sigma}{2}}, \end{aligned}$$

which proves (4.25). Case (ii): $\hat{m}(s) > 1$ for $s \ge s_0$. Then by (4.26) and the monotonicity of $\hat{m}$,

$$s_*^{\frac{3+2\sigma}{6}}(\alpha+\hat{\zeta}(s_*))^{\frac{3+2\sigma}{2}} \le \frac{3}{4\pi}\, c_6^{\frac{3+2\sigma}{2}}\alpha^{3/2}\, s_*^{-\frac{3-2\sigma}{6}}\hat{m}(2s_*) < \left(\frac{\alpha}{20}\right)^{\frac{3+2\sigma}{2}}\hat{m}(2s_*)^{\frac{3+2\sigma}{2}},$$

where we used (4.21), $\hat{m}(2s_*) > 1$ as well as $\frac{3+2\sigma}{2} > 1$. Therefore (4.25) holds in all cases, which in turn proves that $R_* = \infty$ is impossible.

Therefore we must have $R_* < \infty$. Since both ρ_* and m_* are monotone, the limits $\rho_*(R_*) := \lim_{r\to R_*}\rho_*(r) \in [0,\eta]$ and $m_*(R_*) = \lim_{r\to R_*} m_*(r)$ do exist; note that $\frac{2m_*(r)}{r} < 1$ for $r \in [0,R_*[$ implies that m_* is bounded on $[0,R_*]$ and $m_*(R_*) \le R_*/2$.

If $\rho_*(R_*) > 0$ and $\frac{2m_*(R_*)}{R_*} < 1$, then an argument similar to the one in the proof of Lemma 4.3 shows that the solution ρ_* can be slightly extended beyond R_* to a solution $\tilde{\rho}$ on some interval $[0,R_*+\varepsilon[$ such that $0 < \tilde{\rho}(r) \le \eta$ and $\frac{2\tilde{m}(r)}{r} < 1$ for $r \in [0,R_*+\varepsilon[$, which however contradicts the fact that ρ is maximal.

The next case to consider is $R_* < \infty$ and $\frac{2m_*(R_*)}{R_*} = 1$. Let $r_* \in]0,R_*[$ be such that $\frac{2m_*(r_*)}{r_*} \ge \frac{1}{2}$. ¿From (4.14) we obtain

$$\frac{d}{dr}\ln(-\zeta_*(r)) = \frac{\zeta_*'(r)}{\zeta_*(r)} = \frac{m_*(r)}{r^2}\,\frac{1}{1-\frac{2m_*(r)}{r}} + \kappa^{-1}\frac{r}{1-\frac{2m_*(r)}{r}}\,\frac{G(-\zeta_*(r))}{(-\zeta_*(r))}$$

for $r \in [0,R_*[$, whence $\frac{G(-\zeta_*(r))}{(-\zeta_*(r))} \ge 0$ in conjunction with (3.17) yields

$$\alpha \ge (-\zeta_*(R_*)) \ge (-\zeta_*(r_*))\, e^{\int_{r_*}^{R_*}\frac{m_*(r)}{r(r-2m_*(r))}\,dr} \ge \frac{1}{\sqrt{2}}\,\alpha\, e^{\int_{r_*}^{R_*}\frac{m_*(r)}{r(r-2m_*(r))}\,dr}.$$

Therefore

$$\frac{1}{R_*}\int_{r_*}^{R_*}\frac{m_*(r)}{r-2m_*(r)}\,dr \le \int_{r_*}^{R_*}\frac{m_*(r)}{r(r-2m_*(r))}\,dr \le \ln\sqrt{2}.$$

Since m_* is monotone, we get $m_*(r) \geq m_*(r_*) \geq \frac{r_*}{4}$ for $r \in [r_*, R_*]$. It follows that

$$\int_{r_*}^{R_*} \frac{dr}{R_* - 2m_*(r)} \leq \int_{r_*}^{R_*} \frac{dr}{r - 2m_*(r)} \leq 4 \ln \sqrt{2}\, \frac{R_*}{r_*}. \tag{4.27}$$

Due to $m_*'(r) = 4\pi r^2 \rho_*(r) > 0$ for $r \in]0, R_*[$, the function $[r_*, R_*] \ni r \mapsto s(r) = m_*(r) \in [m_*(r_*), m_*(R_*)] = [s_*, S_*]$ is invertible. Thus changing variables as $s = m_*(r)$, $ds = m_*'(r)\, dr$, we deduce from (4.27)

$$\frac{1}{4\pi R_*^2} \int_{s_*}^{S_*} \frac{ds}{\rho_*(r(s))(R_* - 2s)} \leq \int_{s_*}^{S_*} \frac{ds}{4\pi r^2 \rho_*(r(s))(R_* - 2s)} \leq 4 \ln \sqrt{2}\, \frac{R_*}{r_*}.$$

As $S_* = m_*(R_*) = R_*/2$ we get

$$\int_{s_*}^{S_*} \frac{ds}{\rho_*(r(s))(S_* - s)} \leq 32\pi \ln \sqrt{2}\, \frac{R_*^3}{r_*}.$$

Now $0 < \rho_*(r) \leq \eta$ for $r \in [0, R_*]$. Therefore we arrive at the contradiction

$$\int_{s_*}^{S_*} \frac{ds}{S_* - s} \leq 32\pi \ln \sqrt{2}\, \frac{R_*^3}{r_*}\, \eta.$$

If we summarize the above cases so far, we can deduce that in fact we must have $R_* < \infty$, $\rho_*(R_*) = 0$ and $\frac{2m_*(R_*)}{R_*} < 1$. Since we know that (4.14) holds, this implies that $l(r) = 0$ for l from (3.10). According to Remark 3.1, this means that m_* verifies the Euler-Lagrange equation. For the last assertion about the monotonicity of ρ_*, this follows from Lemma 4.1(b). □

Remark 4.5 (a) In [28] it is asserted that for all (sufficiently small) ADM masses M there is a static solution of this mass M. Note that so far this is not included in our results, since both the end of the support R_* of ρ_* and $M = 4\pi \int_0^{R_*} r^2 \rho_*(r)\, dr$ are not explicit.
(b) One might wonder what goes wrong with the whole argument in the massless case. Typically massless (small) solutions are not expected to have compact support, cf. [4]. Also if G from (3.3) would be replaced by

$$\tilde{G}(\varepsilon) = \int_0^\infty \xi^2 \left(\alpha - \varepsilon \xi\right)_+^{\sigma+1} d\xi, \quad \varepsilon \in \mathbb{R}, \tag{4.28}$$

then the estimates for G and $\tilde{G}$ differ drastically and the arguments in this work break down.
(c) The bound $2m(r)/r < 1$ obtained in the proof can be improved once we have a static solution. Indeed, in [1] it is shown that $2m(r)/r < 8/9$ when $p \geq 0$ and $p + 2p_T \leq \rho$. Both conditions hold in this case. ♢

5 Proof of Theorem 3.4

We will show that a solution to the Euler-Lagrange equation (3.9) yields a static solution. First we define a set of functions Ξ_r that is essential for our approach. Let $r > 0$ be fixed and put

$$\Xi_r = \{\chi = \chi(w,\beta) \in L^{1+1/\sigma}(\mathbb{R}\times[0,\infty[) : \chi \geq 0, \chi(w,\beta) = 0 \text{ for } \sqrt{1+w^2+\beta/r^2} \geq 2\}.$$

Then $\Xi_r \subset L^{1+1/\sigma}(\mathbb{R}\times[0,\infty[)$ is closed and convex; the number "2" is chosen such that the support is sufficiently large. The size of the support is dictated by the estimate of ζ in Lemma 3.2. Now consider the functionals

$$H(\chi) = \int_0^\infty d\beta \int_{\mathbb{R}} dw\, (\hat{\Phi}(\chi) - \alpha\chi), \quad F(\chi, r, a) = \int_0^\infty d\beta \int_{\mathbb{R}} dw\, \sqrt{1+w^2+\beta/r^2}\,\chi - \frac{r^2}{\pi}\,a, \quad (5.1)$$

for $r > 0$, functions $\chi = \chi(w,\beta) \in \Xi_r$ and $a > 0$.

Lemma 5.1 *For $r > 0$ and $a > 0$ there is a unique function $\chi = \chi(w,\beta;r,a) \in \Xi_r$ such that*

$$\inf_{\chi\in\Xi_r} \{H(\chi) : F(\chi,r,a) = 0\} = H(\chi(\cdot,\cdot;r,a)) \quad (5.2)$$

and

$$F(\chi(\cdot,\cdot;r,a),r,a) = 0. \quad (5.3)$$

For $r > 0$ and $a = 0$ this unique function is $\chi(w,\beta;r,0) = 0$.

Proof: Define $I = \inf_{\chi\in\Xi_r} \{H(\chi) : F(\chi,r,a) = 0\}$ and let $\chi \in \Xi_r$ be such that $F(\chi,r,a) = 0$. Then

$$\begin{aligned} H(\chi) &= \int_0^\infty d\beta \int_{\mathbb{R}} dw \left(\frac{\sigma}{\sigma+1}\,\chi^{1+1/\sigma} - \alpha\chi\right) \geq -\alpha \int_0^\infty d\beta \int_{\mathbb{R}} dw\,\chi \\ &\geq -\alpha \int_0^\infty d\beta \int_{\mathbb{R}} dw\,\sqrt{1+w^2+\beta/r^2}\,\chi = -\alpha\,\frac{r^2}{\pi}\,a. \end{aligned}$$

Thus $I \geq -\alpha\frac{r^2}{\pi}a$ is finite. The function $x \mapsto x^{1+1/\sigma}$ is strictly convex and the part of H containing $\alpha\chi$ is linear. Therefore H itself is strictly convex. In addition, H is continuous on $\Xi_r \subset L^{1+1/\sigma}(\mathbb{R}\times[0,\infty[)$, since the $L^1(\mathbb{R}\times[0,\infty[)$-norm on Ξ_r can be controlled in terms of the $L^{1+1/\sigma}(\mathbb{R}\times[0,\infty[)$-norm, due to the supports of functions in Ξ_r being contained in $\{\sqrt{1+w^2+\beta/r^2} \leq 2\}$. Let $(\chi_j) \subset \Xi_r$ be a minimizing sequence for I, i.e., we have $\lim_{j\to\infty} H(\chi_j) = I$ and $F(\chi_j,r,a) = 0$ for $j \in \mathbb{N}$. Then

$$\begin{aligned} \frac{\sigma}{\sigma+1}\int_0^\infty d\beta \int_{\mathbb{R}} dw\,\chi_j^{1+1/\sigma} &= H(\chi_j) + \alpha \int_0^\infty d\beta \int_{\mathbb{R}} dw\,\chi_j \\ &\leq H(\chi_j) + \alpha \int_0^\infty d\beta \int_{\mathbb{R}} dw\,\sqrt{1+w^2+\beta/r^2}\,\chi_j \\ &= H(\chi_j) + \alpha\,\frac{r^2}{\pi}\,a \end{aligned}$$

shows that $(\chi_j) \subset L^{1+1/\sigma}(\mathbb{R} \times [0, \infty[)$ is bounded. Hence we may assume that $\chi_j \rightharpoonup \chi$ weakly in $L^{1+1/\sigma}(\mathbb{R} \times [0, \infty[)$ as $j \to \infty$. Since Ξ_r is weakly closed, we infer that $\chi \in \Xi_r$, and moreover $H(\chi) \leq \liminf_{j\to\infty} H(\chi_j) = I$. For $j \in \mathbb{N}$ we have

$$\int_0^\infty d\beta \int_{\mathbb{R}} dw\, \mathbf{1}_{\{\sqrt{1+w^2+\beta/r^2}\leq 2\}} \sqrt{1 + w^2 + \beta/r^2}\, \chi_j = \frac{r^2}{\pi}\, a. \tag{5.4}$$

Since $\varphi \in L^{1+\sigma}(\mathbb{R} \times [0, \infty[)$ for $\varphi(w, \beta) = \mathbf{1}_{\{\sqrt{1+w^2+\beta/r^2}\leq 2\}} \sqrt{1 + w^2 + \beta/r^2}$, we may pass to the limit $j \to \infty$ in (5.4) to obtain

$$\int_0^\infty d\beta \int_{\mathbb{R}} dw\, \mathbf{1}_{\{\sqrt{1+w^2+\beta/r^2}\leq 2\}} \sqrt{1 + w^2 + \beta/r^2}\, \chi = \frac{r^2}{\pi}\, a,$$

which means that $F(\chi, r, a) = 0$, and hence χ is a minimizer.

For the asserted uniqueness, note that this follows from the strict convexity of H and the convexity of the set $\{\chi \in \Xi_r : F(\chi, r, a) = 0\}$. In fact, if $\chi, \tilde{\chi} \in \Xi_r$ are minimizers, then $F(\chi, r, a) = F(\tilde{\chi}, r, a) = 0$ implies that $F((\chi + \tilde{\chi})/2, r, a) = 0$ and $(\chi + \tilde{\chi})/2 \in \Xi_r$. If we had $\chi \neq \tilde{\chi}$, then $I \leq H((\chi + \tilde{\chi})/2) < H(\chi)/2 + H(\tilde{\chi})/2 = I$, which is impossible. □

Now we are in a position to introduce the desired static solution. Let $R_* > 0$, m_* and ρ_* be given by Proposition 3.3 and put $M = m_*(R_*)$. Also recall that by construction $0 \leq \frac{2m_*(r)}{r} < 1$ as well as $0 \leq \rho_*(r) \leq \eta$ for $r \in [0, R_*]$. Another preliminary observation is that

$$\begin{aligned} -\mathcal{G}'(4\pi\kappa\eta) &= \alpha - (\mathcal{G}'(4\pi\kappa\eta) - \mathcal{G}'(0)) = \alpha - \int_0^{4\pi\kappa\eta} \mathcal{G}''(\lambda)\, d\lambda \\ &\geq \alpha - c_4\, \alpha^{\frac{3}{3+2\sigma}} \int_0^{4\pi\kappa\eta} \lambda^{-\frac{1+2\sigma}{3+2\sigma}}\, d\lambda = \alpha - \Big(\frac{3}{2} + \sigma\Big) c_4 (4\pi\kappa)^{\frac{2}{3+2\sigma}}\, \alpha^{\frac{3}{3+2\sigma}}\, \eta^{\frac{2}{3+2\sigma}} \geq \frac{1}{\sqrt{2}}\, \alpha, \end{aligned}$$

where we have used (3.14) and (3.15). Define

$$f_*(r, w, \beta) = \chi(w, \beta; r, \rho_*(r)), \quad \lambda_*(r) = -\frac{1}{2} \ln\Big(1 - \frac{2m_*(r)}{r}\Big), \tag{5.5}$$

where χ is the function obtained in Lemma 5.1. Then $m_*'(r) = 4\pi r^2 \rho_*(r)$ and $e^{-2\lambda_*(r)} = 1 - \frac{2m_*(r)}{r}$ shows that (2.2) is satisfied. Also $\lim_{r\to 0} \frac{m_*(r)}{r} = 0$ and $m_*(\infty) = M$ imply that $\lambda_*(0) = \lambda_*(\infty) = 0$. To specify μ_*, note first that

$$p_*(r) = \int \frac{w^2}{\sqrt{1 + v^2}}\, f_*\, dv = \frac{\pi}{r^2} \int_0^\infty d\beta \int_{\mathbb{R}} dw\, \frac{w^2}{\sqrt{1 + w^2 + \beta/r^2}}\, f_*(r, w, \beta) \tag{5.6}$$

can already be calculated from f_*. Thus it is possible to introduce μ_* by requiring that

$$\mu_*'(r) = e^{2\lambda_*(r)} \Big(\frac{m_*(r)}{r^2} + 4\pi r p_*(r)\Big), \quad \mu_*(\infty) = 0. \tag{5.7}$$

Then (2.3) is straightforward to verify, and

$$\frac{d}{dr}\, e^{\mu_*(r)} = e^{\mu_*(r)+2\lambda_*(r)} \Big(\frac{m_*(r)}{r^2} + 4\pi r\, p_*(r) \Big) \tag{5.8}$$

holds. Also

$$0 = F(f_*(r,\cdot,\cdot), r, \rho_*(r)) = \int_0^\infty d\beta \int_{\mathbb{R}} dw\, \sqrt{1+w^2+\beta/r^2}\, f_*(r,w,\beta) - \frac{r^2}{\pi}\, \rho_*(r)$$

by (5.3) and the definition of F. As a consequence,

$$\rho_{f_*} = \int dv\, \sqrt{1+v^2}\, f_* = \frac{\pi}{r^2} \int_0^\infty d\beta \int_{\mathbb{R}} dw\, \sqrt{1+w^2+\beta/r^2}\, f_* = \rho_*,$$

and in particular $4\pi \int_0^\infty r^2\, \rho_{f_*}(r)\, dr = M$. Also $\rho_*(r) = 0$ for $r \in [R_*, \infty[$ in conjunction with (5.5) and Lemma 5.1 yields $f_*(r,w,\beta) = 0$ for $r \in [R_*, \infty[$. We also note that since $m_*(r) = M$, $p_*(r) = 0$ and $\lambda_*(r) = -\frac{1}{2} \ln(1 - \frac{2M}{r})$ for $r \geq R_*$, it follows from (5.7) through integration that

$$-\mu_*(R_*) = M \int_{R_*}^\infty \frac{dr}{r^2 - 2Mr} = -\frac{1}{2}\, \ln\Big(1 - \frac{2M}{R_*}\Big). \tag{5.9}$$

It remains to see that f_* has the desired form, which is the main part of the argument. We start with a few observations, some of which are close to what has been attempted in [28].

Lemma 5.2 *For $r > 0$, $\chi \in \Xi_r$, $a > 0$ and $\varepsilon \in [\frac{1}{\sqrt{2}}\, \alpha, \alpha]$ let*

$$\begin{aligned} h(\chi, r, a, \varepsilon) &= H(\chi) + \varepsilon F(\chi, r, a) \\ &= \int_0^\infty d\beta \int_{\mathbb{R}} dw\, (\hat{\Phi}(\chi) - (\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2})\, \chi) - \varepsilon\, \frac{r^2}{\pi}\, a \end{aligned} \tag{5.10}$$

and

$$\tilde{h}(r, a, \varepsilon) = \inf_{\chi \in \Xi_r} h(\chi, r, a, \varepsilon).$$

Then

$$\tilde{h}(r, a, \varepsilon) = -\Psi(\varepsilon, r) - \varepsilon\, \frac{r^2}{\pi}\, a \tag{5.11}$$

at fixed (r, a, ε), and the infimum defining $\tilde{h}$ is attained at $\chi_* = \chi_*(\cdot, \cdot; r, a, \varepsilon) \in \Xi_r$ given by

$$\chi_*(w, \beta; r, a, \varepsilon) = (\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2})_+^\sigma = \phi(\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2}), \tag{5.12}$$

i.e., we have

$$h(\chi_*(\cdot, \cdot; r, a, \varepsilon), r, a, \varepsilon) = \tilde{h}(r, a, \varepsilon). \tag{5.13}$$

Proof: To establish (5.11), let first $\chi = \chi(w,\beta) \in \Xi_r$ be arbitrary. Then

$$(\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2})\,\chi(w,\beta) - \hat{\Phi}(\chi(w,\beta)) \le \hat{\hat{\Phi}}(\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2}),$$

and hence

$$h(\chi,r,a,\varepsilon) \ge -\int_0^\infty d\beta \int_{\mathbb{R}} dw\, \hat{\hat{\Phi}}(\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2}) - \varepsilon\,\frac{r^2}{\pi}\,a,$$

from where we deduce that also

$$\tilde{h}(r,a,\varepsilon) \ge -\int_0^\infty d\beta \int_{\mathbb{R}} dw\, \hat{\hat{\Phi}}(\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2}) - \varepsilon\,\frac{r^2}{\pi}\,a.$$

For the converse, for every (w,β) consider the function

$$\varphi(s) = \hat{\Phi}(s) - (\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2})s, \quad s \in \mathbb{R},$$

and recall (3.1). If $\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2} \le 0$, then the minimum of φ is attained at $s_* = 0$, where $\varphi(s_*) = 0$. Secondly, if $\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2} \ge 0$, then the minimum of φ is attained at $s_* = (\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2})^\sigma$, where $\varphi(s_*) = -\frac{1}{\sigma+1}\,(\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2})^{\sigma+1}$. Thus both cases can be summarized as follows: the minimum of φ is attained at $s_* = (\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2})_+^\sigma$, where $\varphi(s_*) = -\frac{1}{\sigma+1}\,(\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2})_+^{\sigma+1}$. Let χ_* be defined as in (5.12), and we drop the variables (r,a,ε) from its arguments to simplify notation. Then $\chi_* \in \Xi_r$, since $0 \le \chi_*(w,\beta) \le \alpha^\sigma$ shows that χ_* is bounded, and it has compact support, so $\chi_* \in L^{1+1/\sigma}(\mathbb{R}\times[0,\infty[)$. Furthermore, if $\sqrt{1+w^2+\beta/r^2} \ge 2$, then $\varepsilon\sqrt{1+w^2+\beta/r^2} \ge 2\,\frac{1}{\sqrt{2}}\,\alpha > \alpha$ and hence $\chi_*(w,\beta) = 0$. Therefore indeed $\chi_* \in \Xi_r$, and $s_* = \chi_*(w,\beta)$. Also

$$\begin{aligned}
\hat{\Phi}(\chi_*(w,\beta)) - (\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2})\,\chi_*(w,\beta) &= \inf_{s\in\mathbb{R}}\,(\hat{\Phi}(s) - (\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2})s) \\
&= -\sup_{s\in\mathbb{R}}\,(s(\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2}) - \hat{\Phi}(s)) \\
&= -\hat{\hat{\Phi}}(\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2}).
\end{aligned}$$

It follows that

$$\begin{aligned}
\tilde{h}(r,a,\varepsilon) &\le h(\chi_*,r,a,\varepsilon) \\
&= \int_0^\infty d\beta \int_{\mathbb{R}} dw\,(\hat{\Phi}(\chi_*) - (\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2})\,\chi_*) - \varepsilon\,\frac{r^2}{\pi}\,a \\
&= -\int_0^\infty d\beta \int_{\mathbb{R}} dw\, \hat{\hat{\Phi}}(\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2}) - \varepsilon\,\frac{r^2}{\pi}\,a,
\end{aligned}$$

This finishes the proof of (5.11), taking into account that $\hat{\hat{\Phi}} = \Phi$ and the definition of Ψ in (3.2). The fact that the infimum is attained at χ_* is a consequence of the above argument, since this function realizes the pointwise minimum of the integrand $\hat{\Phi}(s) - (\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2})s$. □

Lemma 5.3 *Let $r > 0$ and $a \in [0, \eta]$. Then we have*

$$\sup_{\varepsilon \in [\frac{1}{\sqrt{2}}\,\alpha, \alpha]} \tilde{h}(r, a, \varepsilon) = \sup_{\varepsilon \in \mathbb{R}} \tilde{h}(r, a, \varepsilon),$$

and the supremum is uniquely attained at $\varepsilon_* = (G')^{-1}(-4\pi\kappa\, a) \in [\frac{1}{\sqrt{2}}\,\alpha, \alpha]$, taking the value

$$\tilde{h}(r, a, \varepsilon_*) = \sup_{\varepsilon \in \mathbb{R}} \tilde{h}(r, a, \varepsilon) = \hat{\Psi}\Big(- \frac{r^2}{\pi}\, a, r\Big) = \frac{4r^2}{\sigma + 1}\, \hat{G}(-4\pi\kappa a). \tag{5.14}$$

Proof: From (5.11) and (3.4) we obtain

$$\tilde{h}(r, a, \varepsilon) = -\Psi(\varepsilon, r) - \varepsilon\, \frac{r^2}{\pi}\, a = -\frac{4r^2}{\sigma + 1}\, \Big(G(\varepsilon) + \varepsilon\, \frac{\sigma + 1}{4\pi}\, a\Big) = -\frac{4r^2}{\sigma + 1}\, (G(\varepsilon) + 4\pi\kappa\, a\varepsilon),$$

and hence

$$\sup_{\varepsilon \in I} \tilde{h}(r, a, \varepsilon) = -\frac{4r^2}{\sigma + 1}\, \inf_{\varepsilon \in I} (G(\varepsilon) + 4\pi\kappa\, a\, \varepsilon)$$

for $I \subset \mathbb{R}$. Since $G(\varepsilon) = \infty$ for $\varepsilon \in]-\infty, 0]$, no infimum is attained at such ε. If $\varepsilon \geq \alpha$, then $G(\varepsilon) = 0$, and hence $G(\varepsilon) + 4\pi\kappa\, a\, \varepsilon \geq G(\alpha) + 4\pi\kappa\, a\, \alpha$. For $\varepsilon \in]0, \alpha[$, the function $\varphi(\varepsilon) = G(\varepsilon) + 4\pi\kappa\, a\, \varepsilon$ has a minimum, where $\varphi'(\varepsilon) = 0$, which is at $\varepsilon_* = (G')^{-1}(-4\pi\kappa a)$. The function $(G')^{-1}$ is increasing in $]-\infty, 0]$, and $0 \leq a \leq \eta$, whence we deduce that $\varepsilon_* \geq (G')^{-1}(-4\pi\kappa\eta)$. But $\mathcal{G}(s) = \hat{G}(-s)$ by definition, thus $\mathcal{G}'(s) = -\hat{G}'(-s)$ and (7.14) yields $\varepsilon_* \geq (G')^{-1}(-4\pi\kappa\eta) = \hat{G}'(-4\pi\kappa\eta) = -\mathcal{G}'(4\pi\kappa\eta) \geq \frac{1}{\sqrt{2}}\,\alpha$. This completes the proof of the lemma. □

Lemma 5.4 *Let $r > 0$, $a \in]0, \eta[$ and $\varepsilon_* = (G')^{-1}(-4\pi\kappa a)$. Then we have*

$$F(\chi_*(\cdot, \cdot; r, a, \varepsilon_*), r, a) = 0.$$

Proof: At $r > 0$ and $a \in]0, \eta[$ fixed we write $\chi_*(\varepsilon) = \chi_*(\varepsilon)(w, \beta) = \chi_*(w, \beta; r, a, \varepsilon)$. Then, by Lemmas 5.2 and 5.3, the function

$$\varphi(\varepsilon) = h(\chi_*(\varepsilon), r, a, \varepsilon) = \tilde{h}(r, a, \varepsilon) = -\frac{4r^2}{\sigma + 1}\, (G(\varepsilon) + 4\pi\kappa\, a\varepsilon)$$

has a minimum at $\varepsilon_* = (G')^{-1}(-4\pi\kappa a) \in]\frac{1}{\sqrt{2}}\,\alpha, \alpha[$, and thus $\varphi'(\varepsilon_*) = 0$. To calculate

$$\frac{d}{d\varepsilon}\, h(\chi_*(\varepsilon), r, a, \varepsilon)\Big|_{\varepsilon = \varepsilon_*},$$

we can invoke (5.10), $\chi_* \geq 0$ and (3.1) to obtain

$$\begin{aligned} \varphi(\varepsilon) &= h(\chi, r, a, \varepsilon) \\ &= \int_0^\infty d\beta \int_{\mathbb{R}} dw\, \Big(\frac{\sigma}{\sigma + 1}\, \chi_*(\varepsilon)(w, \beta)^{1+1/\sigma} - (\alpha - \varepsilon\sqrt{1 + w^2 + \beta/r^2})\, \chi_*(\varepsilon)(w, \beta)\Big) - \varepsilon\, \frac{r^2}{\pi}\, a. \end{aligned}$$

As the support of $\chi_*(\varepsilon)$ is contained in $\{\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2} \geq 0\}$, we may rewrite this as

$$\begin{aligned}
\varphi(\varepsilon) &= \int_0^\infty d\beta \int_{\mathbb{R}} dw\, \mathbf{1}_{\{\alpha-\varepsilon\sqrt{1+w^2+\beta/r^2}\geq 0\}} \Big(\frac{\sigma}{\sigma+1}\, \chi_*(\varepsilon)(w,\beta)^{1+1/\sigma} \\
&\qquad\qquad - (\alpha - \varepsilon\sqrt{1+w^2+\beta/r^2})\, \chi_*(\varepsilon)(w,\beta) \Big) - \varepsilon\, \frac{r^2}{\pi}\, a \\
&= 2r^2 \int_0^\infty dy\, y \int_{\mathbb{R}} dw\, \mathbf{1}_{\{\alpha-\varepsilon\sqrt{1+w^2+y^2}\geq 0\}} \Big(\frac{\sigma}{\sigma+1}\, \chi_*(\varepsilon)(w, r^2y^2)^{1+1/\sigma} \\
&\qquad\qquad - (\alpha - \varepsilon\sqrt{1+w^2+y^2})\, \chi_*(\varepsilon)(w, r^2y^2) \Big) - \varepsilon\, \frac{r^2}{\pi}\, a \\
&= 2r^2 \int_0^\pi d\theta \sin\theta \int_0^{\sqrt{\frac{\alpha^2}{\varepsilon^2}-1}} d\xi\, \xi^2 \Big(\frac{\sigma}{\sigma+1}\, \lambda_*(\varepsilon)(\theta,\xi)^{1+1/\sigma} \\
&\qquad\qquad - (\alpha - \varepsilon\sqrt{1+\xi^2})\, \lambda_*(\varepsilon)(\theta,\xi) \Big) - \varepsilon\, \frac{r^2}{\pi}\, a,
\end{aligned}$$

where

$$\lambda_*(\varepsilon)(\theta,\xi) = \chi_*(\varepsilon)(\xi\cos\theta, r^2\xi^2\sin^2\theta) = (\alpha - \varepsilon\sqrt{1+\xi^2})^\sigma$$

by (5.12) on the domain of integration, and we have used the changes of variables $y^2 = \beta/r^2$, $2y\,dy = d\beta/r^2$ and thereafter $(w,y) = \xi(\cos\theta, \sin\theta)$, $|\det d(w,y)/d(\theta,\xi)| = \xi$. Noting that $\lambda_*(\varepsilon)(\theta, \sqrt{\frac{\alpha^2}{\varepsilon^2} - 1}) = 0$, we can thus differentiate φ close to ε_* to get

$$\begin{aligned}
\varphi'(\varepsilon) &= 2r^2 \int_0^\pi d\theta \sin\theta \int_0^{\sqrt{\frac{\alpha^2}{\varepsilon^2}-1}} d\xi\, \xi^2 \Big(\lambda_*(\varepsilon)(\theta,\xi)^{1/\sigma} - (\alpha - \varepsilon\sqrt{1+\xi^2}) \Big) \lambda_*'(\varepsilon)(\theta,\xi) \\
&\quad + 2r^2 \int_0^\pi d\theta \sin\theta \int_0^{\sqrt{\frac{\alpha^2}{\varepsilon^2}-1}} d\xi\, \xi^2 \sqrt{1+\xi^2}\, \lambda_*(\varepsilon)(\theta,\xi) - \frac{r^2}{\pi}\, a.
\end{aligned}$$

Since $\lambda_*(\varepsilon)(\theta,\xi)^{1/\sigma} = \alpha - \varepsilon\sqrt{1+\xi^2}$, this simplifies to

$$\varphi'(\varepsilon) = 2r^2 \int_0^\pi d\theta \sin\theta \int_0^{\sqrt{\frac{\alpha^2}{\varepsilon^2}-1}} d\xi\, \xi^2 \sqrt{1+\xi^2}\, \lambda_*(\varepsilon)(\theta,\xi) - \frac{r^2}{\pi}\, a.$$

Thus if we undo the transformations, it is found that

$$\begin{aligned}
\varphi'(\varepsilon) &= \int_0^\infty d\beta \int_{\mathbb{R}} dw\, \mathbf{1}_{\{\alpha-\varepsilon\sqrt{1+w^2+\beta/r^2}\geq 0\}} \sqrt{1+w^2+\beta/r^2}\, \chi_*(\varepsilon)(w,\beta) - \frac{r^2}{\pi}\, a \\
&= \int_0^\infty d\beta \int_{\mathbb{R}} dw\, \sqrt{1+w^2+\beta/r^2}\, \chi_*(\varepsilon)(w,\beta) - \frac{r^2}{\pi}\, a \\
&= F(\chi_*(\varepsilon), r, a).
\end{aligned}$$

As a consequence,

$$0 = \varphi'(\varepsilon_*) = F(\chi_*(\varepsilon_*), r, a),$$

as was to be shown. $\square$

Corollary 5.5 *Let $r > 0$, $a \in]0, \eta[$ and $\varepsilon_* = (G')^{-1}(-4\pi\kappa a)$. Then we have*

$$\chi_*(\cdot, \cdot; r, a, \varepsilon_*) = \chi(\cdot, \cdot; r, a).$$

In particular,

$$f_*(r, w, \beta) = \phi(\alpha - \tilde{\varepsilon}(r)\sqrt{1 + w^2 + \beta/r^2}) \tag{5.15}$$

for $\tilde{\varepsilon}(r) = (G')^{-1}(-4\pi\kappa\rho_*(r))$.

Proof: We continue to denote $\chi_*(\varepsilon_*) = \chi_*(\varepsilon_*)(w, \beta) = \chi_*(w, \beta; r, a, \varepsilon_*)$. Then $\chi_*(\varepsilon_*) \in \Xi_r$ by Lemma 5.2 and $F(\chi_*(\varepsilon_*), r, a) = 0$ due to Lemma 5.4. Furthermore, by the various definitions,

$$\begin{aligned} H(\chi_*(\varepsilon_*)) &= H(\chi_*(\varepsilon_*)) + \varepsilon_* F(\chi_*(\varepsilon_*), r, a) = h(\chi_*(\varepsilon_*), r, a, \varepsilon_*) = \tilde{h}(r, a, \varepsilon_*) \\ &= \inf_{\chi \in \Xi_r} h(\chi, r, a, \varepsilon_*) \leq h(\chi(\cdot, \cdot; r, a), r, a, \varepsilon_*) \\ &= H(\chi(\cdot, \cdot; r, a)) + \varepsilon_* F(\chi(\cdot, \cdot; r, a), r, a) = H(\chi(\cdot, \cdot; r, a)). \end{aligned}$$

¿From the uniqueness of the minimizer (Lemma 5.1) we thus deduce that $\chi_*(\varepsilon_*) = \chi(\cdot, \cdot; r, a)$. For (5.15) it suffices to recall (5.5) and to notice that here we take $a = \rho_*(r)$, $\rho_* \leq \eta$ is strictly decreasing on $[0, R_*]$ and such that $\rho_*(R_*) = 0$ by Proposition 3.3, whence $\rho_*(r) \in]0, \eta[$ for $r \in]0, R_*[$. $\square$

Corollary 5.6 *Let $r > 0$ and $a \in]0, \eta[$. Then*

$$\hat{\Psi}\Big(-\frac{r^2}{\pi}\,a, r\Big) = H(\chi(\cdot, \cdot; r, a)).$$

Proof: Define $\varepsilon_* = (G')^{-1}(-4\pi\kappa a)$ as before. Then $\chi_*(\cdot, \cdot; r, a, \varepsilon_*) = \chi(\cdot, \cdot; r, a)$ by Corollary 5.5, and furthermore $F(\chi(\cdot, \cdot; r, a), r, a) = F(\chi_*(\cdot, \cdot; r, a, \varepsilon_*), r, a) = 0$ by Lemma 5.4. Therefore (5.14) and (5.13) yield

$$\begin{aligned} \hat{\Psi}\Big(-\frac{r^2}{\pi}\,a, r\Big) &= \tilde{h}(r, a, \varepsilon_*) = h(\chi_*(\cdot, \cdot; r, a, \varepsilon_*), r, a, \varepsilon_*) = h(\chi(\cdot, \cdot; r, a), r, a, \varepsilon_*) \\ &= H(\chi(\cdot, \cdot; r, a)) + \varepsilon_* F(\chi(\cdot, \cdot; r, a), r, a) = H(\chi(\cdot, \cdot; r, a)), \end{aligned}$$

as claimed. $\square$

We still need to verify that the argument on the right-hand side of (5.15) is a function of the energy $E_* = e^{\mu_*(r)}\sqrt{1 + v^2} = e^{\mu_*(r)}\sqrt{1 + w^2 + \beta/r^2}$. For this we have to relate $\tilde{\varepsilon}(r)$ to $e^{\mu_*(r)}$, and here the Euler-Lagrange equation (3.9) for m_* enters in a crucial way; we will use it in the form $l(r) = 0$ for $r \in [0, R_*]$, cf. Remark 3.1. Hence it follows from (3.10) and (3.4) that

$$\begin{aligned} 0 &= -\zeta'(r) + \frac{m_*(r)}{r^2}\,\frac{1}{1-\frac{2m_*(r)}{r}}\,\zeta(r) - \frac{1}{\kappa}\,\frac{r}{1-\frac{2m_*(r)}{r}}\,G(-\zeta(r)) \\ &= -\zeta'(r) + \frac{m_*(r)}{r^2}\,\frac{1}{1-\frac{2m_*(r)}{r}}\,\zeta(r) - \frac{4\pi^2}{r^2}\,\frac{r}{1-\frac{2m_*(r)}{r}}\,\Psi(-\zeta(r), r) \end{aligned} \tag{5.16}$$

for $\zeta(r) = \mathcal{G}'(\kappa \frac{m_*'(r)}{r^2}) = \mathcal{G}'(4\pi\kappa\rho_*(r)) = -\hat{G}'(-4\pi\kappa\rho_*(r))$. ¿From (7.14) we deduce that

$$\tilde{\varepsilon}(r) = (G')^{-1}(-4\pi\kappa\rho_*(r)) = \hat{G}'(-4\pi\kappa\rho_*(r)), \tag{5.17}$$

which shows that $\zeta(r) = -\tilde{\varepsilon}(r)$. Therefore (5.16) comes down to

$$0 = \tilde{\varepsilon}'(r) - \frac{1}{r-2m_*(r)}\,\frac{m_*(r)}{r}\,\tilde{\varepsilon}(r) - \frac{4\pi^2}{r-2m_*(r)}\,\Psi(\tilde{\varepsilon}(r), r).$$

Since $e^{2\lambda_*(r)} = \frac{r}{r-2m_*(r)}$, we get

$$\tilde{\varepsilon}'(r) = e^{2\lambda_*(r)}\Big(\frac{m_*(r)}{r^2}\,\tilde{\varepsilon}(r) + \frac{4\pi^2}{r}\,\Psi(\tilde{\varepsilon}(r), r)\Big). \tag{5.18}$$

Lastly, from (3.5), (5.15) and (5.6) we obtain

$$\begin{aligned} \Psi(\tilde{\varepsilon}(r), r) &= \tilde{\varepsilon}(r)\int_0^\infty d\beta \int_{\mathbb{R}} dw\,\frac{w^2}{\sqrt{1+w^2+\beta/r^2}}\,\phi(\alpha - \tilde{\varepsilon}(r)\sqrt{1+w^2+\beta/r^2}) \\ &= \tilde{\varepsilon}(r)\int_0^\infty d\beta \int_{\mathbb{R}} dw\,\frac{w^2}{\sqrt{1+w^2+\beta/r^2}}\,f_*(r,w,\beta) \\ &= \frac{r^2}{\pi}\,\tilde{\varepsilon}(r)\,p_*(r). \end{aligned}$$

Hence (5.18) shows that

$$\tilde{\varepsilon}'(r) = \tilde{\varepsilon}(r)\,e^{2\lambda_*(r)}\Big(\frac{m_*(r)}{r^2} + 4\pi r\,p_*(r)\Big).$$

Comparing this to (5.8), it follows that

$$\frac{d}{dr}\,(e^{-\mu_*(r)}\tilde{\varepsilon}(r)) = 0, \;\; 0 \le r \le R_*,$$

so that $e^{-\mu_*(r)}\tilde{\varepsilon}(r) = c_*$ for some constant c_*. Thus $\tilde{\varepsilon}(r) = c_* e^{\mu_*(r)}$. Next we have from (5.9) that $e^{\mu_*(R_*)} = \sqrt{1-2M/R_*}$. Furthermore, $\tilde{\varepsilon}(r) = \hat{G}'(-4\pi\kappa\rho_*(r))$ by (5.17) together with $\rho_*(R_*) = 0$ and $\hat{G}'(0) = \alpha$ leads to $\tilde{\varepsilon}(R_*) = \alpha$, which in turn yields

$$c_* = \frac{\alpha}{\sqrt{1-\frac{2M}{R_*}}}.$$

Therefore (5.15) finally implies that

$$f_*(r,w,\beta) = \phi\Big(\alpha - \frac{\alpha}{\sqrt{1-\frac{2M}{R_*}}}\, e^{\mu_*(r)}\sqrt{1+w^2+\beta/r^2}\Big),$$

which has the desired form, depending only on E_*. This completes the proof of Theorem 3.4. □

6 Relations to stability

Consider the particle number-Casimir functional $\mathcal{D}$ from (1.1). We begin with a lemma that makes a relation between $\mathcal{D}$ and $\mathcal{L}$, see (1.2).

Lemma 6.1 *Let $\alpha > 0$ be fixed and suppose that $\eta > 0$ is such that (3.15) is verified. Let a static solution (f_*, λ_*, μ_*) be constructed as in Theorem 3.4. Then*

(a) $\mathcal{L}(m_*) = \kappa\,\mathcal{D}(f_*)$;

(b) if (f,λ,μ) is a further (possibly time-dependent) solution so that

$$f(t,r,\cdot,\cdot) \in \Xi_r, \quad \frac{2m(t,r)}{r} < 1, \quad \rho(t,r) \in]0,\eta[, \tag{6.1}$$

for $t \in [0,T]$ and $r \in]0,\infty[$, then $\mathcal{L}(m(t)) \le \kappa\,\mathcal{D}(f(t))$ for $t \in [0,T]$, where $m(t)(r) = m(t,r)$ and $f(t)(r,w,\beta) = f(t,r,w,\beta)$.

Proof: (a) We have $\rho_{f_*} = \rho_*$, $\lambda_{f_*} = \lambda_*$ and $e^{-2\lambda_*(r)} = 1 - \frac{2m_*(r)}{r}$. Hence by (2.1), (5.1), and due to $f_*(r,w,\beta) = 0$ for $r \in [R_*,\infty[$:

$$\begin{aligned}
\mathcal{D}(f_*) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{\lambda_*}\,(\hat\Phi(f_*) - \alpha f_*)\,dx\,dv \\
&= 4\pi^2\int_0^\infty dr\,\sqrt{\frac{r}{r-2m_*(r)}}\int_0^\infty d\beta\int_{\mathbb{R}} dw\,(\hat\Phi(f_*) - \alpha f_*) \\
&= 4\pi^2\int_0^{R_*} dr\,\sqrt{\frac{r}{r-2m_*(r)}}\int_0^\infty d\beta\int_{\mathbb{R}} dw\,(\hat\Phi(f_*) - \alpha f_*) \\
&= 4\pi^2\int_0^{R_*}\sqrt{\frac{r}{r-2m_*(r)}}\,H(f_*(r,\cdot,\cdot))\,dr \\
&= 4\pi^2\int_0^{R_*}\sqrt{\frac{r}{r-2m_*(r)}}\,H(\chi(\cdot,\cdot;r,\rho_*(r)))\,dr.
\end{aligned}$$

Also $\frac{m_*'(r)}{r^2} = 4\pi\rho_*(r) = 0$ for $r \in [R_*, \infty[$. Since $\hat{G}(0) = \mathcal{G}(0) = 0$ by Lemma 7.8, it follows that

$$\begin{aligned} \mathcal{L}(m_*) &= \int_0^\infty L(r, m_*(r), m_*'(r))\, dr \\ &= \int_0^{R_*} \frac{r^{5/2}}{\sqrt{r - 2m_*(r)}}\, \hat{G}\Big(-\kappa\, \frac{m_*'(r)}{r^2}\Big)\, dr \\ &= \frac{\sigma+1}{4} \int_0^{R_*} \sqrt{\frac{r}{r - 2m_*(r)}}\, \hat{\Psi}\Big(-\frac{r^2}{\pi}\, \rho_*(r), r\Big)\, dr, \end{aligned}$$

where we have used (3.6) in the last step. It remains to apply Corollary 5.6 for $a = \rho_*(r)$. (b) In the same way as in (a) we obtain

$$\begin{aligned} \mathcal{D}(f(t)) &= 4\pi^2 \int_0^\infty \sqrt{\frac{r}{r - 2m(r)}}\, H(f(t, r, \cdot, \cdot))\, dr, \\ \mathcal{L}(m(t)) &= \frac{\sigma+1}{4} \int_0^\infty \sqrt{\frac{r}{r - 2m(t, r)}}\, \hat{\Psi}\Big(-\frac{r^2}{\pi}\, \rho(t, r), r\Big)\, dr. \end{aligned}$$

Since $\rho(t, r) \in]0, \eta[$, Corollary 5.6 implies that $\hat{\Psi}(-\frac{r^2}{\pi}\, \rho(t, r), r) = H(\chi(\cdot, \cdot; r, \rho(t, r)))$. By (2.5) we have

$$\frac{r^2}{\pi}\, \rho(t, r) = \int_{\mathbb{R}} \int_0^\infty \sqrt{1 + w^2 + \beta/r^2}\, f(t, r, w, \beta)\, d\beta\, dw,$$

which means that $F(f(t, r, \cdot, \cdot), r, \rho(r)) = 0$. As we are assuming that $f(t, r, \cdot, \cdot) \in \Xi_r$, (5.2) shows that $H(\chi(\cdot, \cdot; r, \rho(t, r))) \le H(f(t, r, \cdot, \cdot))$. Altogether, this yields $\mathcal{L}(m(t)) \le \frac{\sigma+1}{16\pi^2}\, \mathcal{D}(f(t)) = \kappa\, \mathcal{D}(f(t))$ for $t \in [0, T]$. □

Remark 6.2 (a) Note that since $\mathcal{D}$ is constant along solutions, in fact $\mathcal{D}(f(t)) = \mathcal{D}(f(0))$ for $t \in [0, T]$ in part (b).

(b) The hypotheses (6.1) are rather strong (for instance, they imply global existence) and should be considered only as an illustration. In an actual proof of non-linear stability, one would not be able to make global assumptions on bounds for the solution, but rather it would be needed to derive those bounds from some kind of stability estimate; for the relativistic Vlasov-Poisson system this has been accomplished in [15].

(c) We would like to be able to say that f_* minimizes $\mathcal{D}$ over a certain class of functions $\mathcal{F}$,

$$\mathcal{D}(f) \ge \mathcal{D}(f_*), \quad f \in \mathcal{F}. \tag{6.2}$$

In view of Lemma 6.1, for this we could resort to a general method, as for instance outlined in [8, Prop. 1.18]. We write

$$\mathcal{D}(f) - \mathcal{D}(f_*) = [\mathcal{D}(f) - \kappa^{-1}\mathcal{L}(m)] + [\kappa^{-1}\mathcal{L}(m) - \kappa^{-1}\mathcal{L}(m_*)]$$

and we already know that the first [...] is non-negative. Thus to establish (6.2), the real issue is to show that m_* minimizes $\mathcal{L}$ over a certain set of functions related to $\mathcal{F}$. In other words, since m_* solves the associated Euler-Lagrange equation, it has to be clarified if this in turn does imply that m_* is a minimizer (or if some condition has to be added). Generally speaking, for some variational problems this is possible, using "Mayer fields"; see [8] or [9, Thm. 4.18]. $\diamondsuit$

7 Appendix

7.1 Properties of G, G', G'' and $\mathcal{G}$

¿From (3.3) recall that

$$G(\varepsilon) = \int_0^\infty \xi^2 \left(\alpha - \varepsilon\sqrt{1+\xi^2}\right)_+^{\sigma+1} d\xi$$

for $\sigma \in]0, \frac{3}{2}[$, where it is understood that $G(\varepsilon) = \infty$ for $\varepsilon \in]-\infty, 0]$. Here $\alpha > 0$ is a parameter. Clearly $G(\varepsilon) = 0$ for $\varepsilon \in [\alpha, \infty[$.

Henceforth we are going to write $A \sim B$ for two functions $A(x) \geq 0$ and $B(x) \geq 0$, if there are constants $c_1, c_2 > 0$ such that $c_1 A(x) \leq B(x) \leq c_2 A(x)$ for all x, where A and B are defined. In other words, in each quantitative estimate (from above or from below) A could be exchanged by B, and vice versa.

Lemma 7.1 *For $a \geq 0$ and $b > -1$ let*

$$\Phi_{a,b}(\theta) = \int_0^\theta \xi^2 \left(\sqrt{1+\xi^2}\right)^a \frac{(\theta^2 - \xi^2)^b}{\left(\sqrt{1+\theta^2} + \sqrt{1+\xi^2}\right)^b} d\xi, \quad \theta \in [0, \infty[. \tag{7.1}$$

Then $\Phi_{a,b}(\theta) \sim \theta^{3+2b}$ for $\theta \in [0, 1]$ and $\Phi_{a,b}(\theta) \sim \theta^{3+b+a}$ for $\theta \in [1, \infty[$.

Proof: To begin with, if $0 \leq \xi \leq \theta$, then

$$\sqrt{1+\theta^2} \leq \sqrt{1+\theta^2} + \sqrt{1+\xi^2} \leq 2\sqrt{1+\theta^2},$$

so that

$$\Phi_{a,b}(\theta) \sim \frac{1}{(1+\theta^2)^{b/2}} \int_0^\theta \xi^2 \left(\sqrt{1+\xi^2}\right)^a (\theta^2 - \xi^2)^b d\xi.$$

Also $\theta \leq \theta + \xi \leq 2\theta$, which results in

$$\Phi_{a,b}(\theta) \sim \frac{\theta^b}{(1+\theta^2)^{b/2}} \int_0^\theta \xi^2 \left(\sqrt{1+\xi^2}\right)^a (\theta - \xi)^b d\xi = \frac{\theta^{3+2b}}{(1+\theta^2)^{b/2}} \int_0^1 s^2 \left(\sqrt{1+\theta^2 s^2}\right)^a (1-s)^b ds,$$

where we have changed variables as $\xi = \theta s$, $d\xi = \theta\, ds$, in the second step; note that the integral is non-singular due to $b > -1$. If $\theta \in [0,1]$, then $\sqrt{1+\theta^2 s^2} \sim 1$ for $s \in [0,1]$ and $1+\theta^2 \sim 1$. It follows that $\Phi_{a,b}(\theta) \sim \theta^{3+2b}$ in this case. On the other hand, if $\theta \in [1,\infty[$, then $1+\theta^2 \sim \theta^2$ and moreover $\theta s \le \sqrt{1+\theta^2 s^2} \le \sqrt{2}\,\theta$ for $s \in [0,1]$ yields $\Phi_{a,b}(\theta) \sim \theta^{3+b+a}$. $\square$

Corollary 7.2 *Let $a \ge 0$ and $b > -1$. There is a constant $C_* > 0$ such that*

$$|\Phi_{a,b}(\theta) - \Phi_{a,b}(\tilde\theta)| \le C_* |\theta - \tilde\theta|, \quad \theta, \tilde\theta \in [0,1].$$

Proof: In the case where $b \ge 0$ we may simply differentiate $\Phi_{a,b}(\theta)$ and bound the derivative. Thus we may assume that $b \in]-1,0[$ to rewrite $\Phi_{a,b}(\theta)$ as

$$\begin{aligned} \Phi_{a,b}(\theta) &= \int_0^\theta \xi^2 (\sqrt{1+\xi^2})^a \frac{(\sqrt{1+\theta^2}+\sqrt{1+\xi^2})^{\hat b}}{(\theta^2-\xi^2)^{\hat b}}\, d\xi \\ &= \theta^{3-2\hat b} \int_0^1 \tau^2 (\sqrt{1+\theta^2\tau^2})^a \frac{(\sqrt{1+\theta^2}+\sqrt{1+\theta^2\tau^2})^{\hat b}}{(1-\tau^2)^{\hat b}}\, d\tau \end{aligned} \tag{7.2}$$

for $\hat b = -b \in]0,1[$, and we used the change of variables $\xi = \theta\tau$, $d\xi = \theta d\tau$; note that $3 - 2\hat b \in]1,3[$. Since

$$\int_0^1 \frac{1}{(1-\tau)^{\hat b}}\, d\tau < \infty$$

is integrable, we can differentiate (7.2) to obtain the expression

$$\begin{aligned} \Phi_{a,b}(\theta) &= (3-2\hat b)\,\theta^{2(1-\hat b)} \int_0^1 \tau^2 (\sqrt{1+\theta^2\tau^2})^a \frac{(\sqrt{1+\theta^2}+\sqrt{1+\theta^2\tau^2})^{\hat b}}{(1-\tau^2)^{\hat b}}\, d\tau \\ &\quad + a\,\theta^{4-2\hat b} \int_0^1 \tau^4 (\sqrt{1+\theta^2\tau^2})^{a-2} \frac{(\sqrt{1+\theta^2}+\sqrt{1+\theta^2\tau^2})^{\hat b}}{(1-\tau^2)^{\hat b}}\, d\tau \\ &\quad + \hat b\,\theta^{4-2\hat b} \int_0^1 \tau^2 (\sqrt{1+\theta^2\tau^2})^a \frac{1}{(\sqrt{1+\theta^2}+\sqrt{1+\theta^2\tau^2})^{1-\hat b}(1-\tau^2)^{\hat b}} \\ &\qquad \times \Big(\frac{1}{\sqrt{1+\theta^2}} + \frac{\tau^2}{\sqrt{1+\theta^2\tau^2}}\Big)\, d\tau, \end{aligned}$$

which is bounded in $\theta \in [0,1]$. $\square$

Lemma 7.3 (Properties of G) *We have*

$$G(\varepsilon) \sim \left\{ \begin{array}{ccl} \alpha^{-3/2}\,(\alpha-\varepsilon)^{5/2+\sigma} & : & \varepsilon \in [\frac{1}{\sqrt 2}\,\alpha, \alpha] \\ \varepsilon^{-3}\,\alpha^{4+\sigma} & : & \varepsilon \in]0, \frac{1}{\sqrt 2}\,\alpha] \end{array} \right. . \tag{7.3}$$

In particular, since $G(\varepsilon) = 0$ *for* $\varepsilon \in [\alpha, \infty[$, $G :]0, \infty[\to \mathbb{R}$ *is continuous. Also* $G(\varepsilon) > 0$ *for* $\varepsilon \in]0, \alpha[$.

Proof: Denoting $\tau = \alpha/\varepsilon$ and $\theta = \sqrt{\tau^2 - 1}$, we see that for $\varepsilon \in]0, \alpha]$, and thus $\tau \in [1, \infty[$ as well as $\theta \in [0, \infty[$,

$$\begin{aligned} G(\varepsilon) &= \int_0^\infty \xi^2 \left(\alpha - \varepsilon\sqrt{1+\xi^2}\right)_+^{\sigma+1} d\xi = \varepsilon^{\sigma+1} \int_0^\infty \xi^2 \left(\tau - \sqrt{1+\xi^2}\right)_+^{\sigma+1} d\xi \\ &= \varepsilon^{\sigma+1} \int_0^{\sqrt{\tau^2-1}} \xi^2 \left(\tau - \sqrt{1+\xi^2}\right)^{\sigma+1} d\xi = \varepsilon^{\sigma+1} \int_0^{\sqrt{\tau^2-1}} \xi^2 \, \frac{\left(\tau^2 - 1 - \xi^2\right)^{\sigma+1}}{\left(\tau + \sqrt{1+\xi^2}\right)^{\sigma+1}} \, d\xi \\ &= \varepsilon^{\sigma+1} \int_0^\theta \xi^2 \, \frac{\left(\theta^2 - \xi^2\right)^{\sigma+1}}{\left(\sqrt{1+\theta^2} + \sqrt{1+\xi^2}\right)^{\sigma+1}} \, d\xi = \varepsilon^{\sigma+1} \Phi_{0,\sigma+1}(\theta), \end{aligned}$$

cf. (7.1). Therefore Lemma 7.1 implies that

$$\begin{aligned} G(\varepsilon) &\sim \varepsilon^{\sigma+1} \begin{cases} \theta^{5+2\sigma} & : \ \theta \in [0,1] \\ \theta^{4+\sigma} & : \ \theta \in [1, \infty[\end{cases} \\ &= \varepsilon^{\sigma+1} \begin{cases} (\tau^2 - 1)^{5/2+\sigma} & : \ \tau \in [1, \sqrt{2}] \\ (\tau^2 - 1)^{2+\sigma/2} & : \ \tau \in [\sqrt{2}, \infty[\end{cases} \\ &= \begin{cases} \varepsilon^{-4-\sigma} \, (\alpha^2 - \varepsilon^2)^{5/2+\sigma} & : \ \varepsilon \in [\frac{1}{\sqrt{2}}\,\alpha, \alpha] \\ \varepsilon^{-3} \, (\alpha^2 - \varepsilon^2)^{2+\sigma/2} & : \ \varepsilon \in]0, \frac{1}{\sqrt{2}}\,\alpha] \end{cases} . \end{aligned}$$

It remains to make use of the facts that $\alpha + \varepsilon \sim \alpha$ for $\varepsilon \in [0, \alpha]$, $\varepsilon \sim \alpha$ for $\varepsilon \in [\frac{1}{\sqrt{2}}\,\alpha, \alpha]$ and $\alpha - \varepsilon \sim \alpha$ for $\varepsilon \in]0, \frac{1}{\sqrt{2}}\,\alpha]$. □

Lemma 7.4 (Properties of G') *We have*

$$G'(\varepsilon) = -(\sigma+1) \int_0^\infty \xi^2 \sqrt{1+\xi^2} \left(\alpha - \varepsilon\sqrt{1+\xi^2}\right)_+^{\sigma} d\xi, \quad \varepsilon \in]0, \alpha], \tag{7.4}$$

and

$$G'(\varepsilon) \sim \begin{cases} -\,\alpha^{-3/2} \, (\alpha - \varepsilon)^{3/2+\sigma} & : \ \varepsilon \in [\frac{1}{\sqrt{2}}\,\alpha, \alpha] \\ -\,\varepsilon^{-4} \, \alpha^{4+\sigma} & : \ \varepsilon \in]0, \frac{1}{\sqrt{2}}\,\alpha] \end{cases} . \tag{7.5}$$

In particular, since $G'(\varepsilon) = 0$ *for* $\varepsilon \in]\alpha, \infty[$, $G' :]0, \infty[\to \mathbb{R}$ *is continuous. Also* $G'(\varepsilon) < 0$ *for* $\varepsilon \in]0, \alpha[$.

Proof: By definition we may write

$$G(\varepsilon) = \int_0^{\sqrt{\frac{\alpha^2}{\varepsilon^2}-1}} \xi^2 \left(\alpha - \varepsilon\sqrt{1+\xi^2}\right)^{\sigma+1} d\xi. \tag{7.6}$$

Differentiating w.r. to ε, the upper integration limit term makes no contribution, since $(\ldots)^{\sigma+1}$ vanishes at this point ξ; thus (7.4) follows. Concerning the asymptotics of $G'(\varepsilon)$, we can follow the same steps as in the proof of Lemma 7.3 to obtain the relation

$$G'(\varepsilon) = -(\sigma+1)\,\varepsilon^\sigma\,\Phi_{1,\sigma}(\theta).$$

Therefore we can invoke Lemma 7.1 to get

$$\begin{aligned}
G'(\varepsilon) &\sim -\varepsilon^\sigma \begin{cases} \theta^{3+2\sigma} & : \theta \in [0,1] \\ \theta^{4+\sigma} & : \theta \in [1,\infty[\end{cases} \\
&= -\varepsilon^\sigma \begin{cases} (\tau^2-1)^{3/2+\sigma} & : \tau \in [1,\sqrt{2}] \\ (\tau^2-1)^{2+\sigma/2} & : \tau \in [\sqrt{2},\infty[\end{cases} \\
&= \begin{cases} -\varepsilon^{-3-\sigma}\,(\alpha^2-\varepsilon^2)^{3/2+\sigma} & : \varepsilon \in [\frac{1}{\sqrt{2}}\,\alpha,\alpha] \\ -\varepsilon^{-4}\,(\alpha^2-\varepsilon^2)^{2+\sigma/2} & : \varepsilon \in]0,\frac{1}{\sqrt{2}}\,\alpha] \end{cases},
\end{aligned}$$

which in turn leads to (7.5). □

Lemma 7.5 (Properties of G'') *We have*

$$G''(\varepsilon) = \sigma(\sigma+1)\int_0^\infty \xi^2(1+\xi^2)\left(\alpha - \varepsilon\sqrt{1+\xi^2}\right)_+^{\sigma-1} d\xi, \quad \varepsilon \in]0,\alpha], \tag{7.7}$$

and

$$G''(\varepsilon) \sim \begin{cases} \alpha^{-3/2}\,(\alpha-\varepsilon)^{1/2+\sigma} & : \varepsilon \in [\frac{1}{\sqrt{2}}\,\alpha,\alpha] \\ \varepsilon^{-5}\,\alpha^{4+\sigma} & : \varepsilon \in]0,\frac{1}{\sqrt{2}}\,\alpha] \end{cases}. \tag{7.8}$$

In particular, since $G''(\varepsilon) = 0$ for $\varepsilon \in]\alpha,\infty[$, $G'' :]0,\infty[\to \mathbb{R}$ is continuous. Also $G''(\varepsilon) > 0$ for $\varepsilon \in]0,\alpha[$.

Proof: Relation (7.7) is derived from (7.4) in the same way as (7.4) is obtained from (7.6); for this it is important that $\sigma > 0$. Also we may rewrite $G''(\varepsilon)$ in the form

$$G''(\varepsilon) = \sigma(\sigma+1)\,\varepsilon^{\sigma-1}\,\Phi_{2,\sigma-1}(\theta). \tag{7.9}$$

Hence Lemma 7.1 implies that

$$
\begin{aligned}
G''(\varepsilon) &\sim \varepsilon^{\sigma-1} \begin{cases} \theta^{1+2\sigma} & : \quad \theta \in [0,1] \\ \theta^{4+\sigma} & : \quad \theta \in [1,\infty[\end{cases} \\
&= \varepsilon^{\sigma-1} \begin{cases} (\tau^2-1)^{1/2+\sigma} & : \quad \tau \in [1,\sqrt{2}] \\ (\tau^2-1)^{2+\sigma/2} & : \quad \tau \in [\sqrt{2},\infty[\end{cases} \\
&= \begin{cases} \varepsilon^{-2-\sigma}\,(\alpha^2-\varepsilon^2)^{1/2+\sigma} & : \quad \varepsilon \in [\frac{1}{\sqrt{2}}\,\alpha,\alpha] \\ \varepsilon^{-5}\,(\alpha^2-\varepsilon^2)^{2+\sigma/2} & : \quad \varepsilon \in]0,\frac{1}{\sqrt{2}}\,\alpha] \end{cases},
\end{aligned}
$$

and this gives (7.8). □

Corollary 7.6 *For every $q \in [\frac{1}{\sqrt{2}},1[$ there exists a constant $C_q > 0$ such that*

$$
|G''(\varepsilon) - G''(\tilde{\varepsilon})| \le C_q\,\alpha^{-(2-\sigma)}\,|\varepsilon-\tilde{\varepsilon}|, \quad \varepsilon,\tilde{\varepsilon} \in \Big[\frac{1}{\sqrt{2}}\,\alpha, q\alpha\Big].
$$

Proof: Due to (7.9) we have $G''(\varepsilon) = \sigma(\sigma+1)\,\varepsilon^{\sigma-1}\,\Phi_{2,\sigma-1}(\theta)$, where

$$
\theta = \sqrt{\frac{\alpha^2}{\varepsilon^2}-1} \in [\theta_q,1], \quad \theta_q = \sqrt{\frac{1}{q^2}-1} \in]0,1].
$$

Then

$$
\begin{aligned}
|\theta-\tilde{\theta}| &= \Big|\sqrt{\frac{\alpha^2}{\varepsilon^2}-1} - \sqrt{\frac{\alpha^2}{\tilde{\varepsilon}^2}-1}\Big| = \frac{\alpha^2}{\theta+\tilde{\theta}}\,\frac{|\varepsilon^2-\tilde{\varepsilon}^2|}{\varepsilon^2\tilde{\varepsilon}^2} \\
&\le \frac{4\alpha^2}{2\theta_q}\,\frac{2q\alpha\,|\varepsilon-\tilde{\varepsilon}|}{\alpha^4} \le \frac{4}{\theta_q}\,\alpha^{-1}\,|\varepsilon-\tilde{\varepsilon}|.
\end{aligned}
$$

Let $C_* > 0$ denote the constant from Corollary 7.2. Since $\Phi_{2,\sigma-1}(0) = 0$ by Lemma 7.1, Corollary 7.2 in particular implies that $|\Phi_{2,\sigma-1}(\theta)| \le C_*\theta \le C_*$ for $\theta \in [0,1]$. Hence, using this Corollary 7.2,

$$
\begin{aligned}
|G''(\varepsilon) - G''(\tilde{\varepsilon})| &\le \sigma(\sigma+1)\,|\Phi_{2,\sigma-1}(\theta)|\,|\varepsilon^{\sigma-1}-\tilde{\varepsilon}^{\sigma-1}| \\
&\quad + \sigma(\sigma+1)\,\tilde{\varepsilon}^{\sigma-1}\,|\Phi_{2,\sigma-1}(\theta) - \Phi_{2,\sigma-1}(\tilde{\theta})| \\
&\le C_*\,\sigma(\sigma+1)|\sigma-1|\,\Big(\frac{\sqrt{2}}{\alpha}\Big)^{2-\sigma}\,|\varepsilon-\tilde{\varepsilon}| + C_*\,\sigma(\sigma+1)\,\tilde{\varepsilon}^{\sigma-1}\,|\theta-\tilde{\theta}| \\
&\le C_*\,\sigma(\sigma+1)\,\Big[|\sigma-1|\,\Big(\frac{\sqrt{2}}{\alpha}\Big)^{2-\sigma} + \frac{4}{\theta_q}\,(\max_{u\in[\frac{1}{\sqrt{2}}\alpha,\alpha]} u^{\sigma-1})\,\alpha^{-1}\Big]\,|\varepsilon-\tilde{\varepsilon}|,
\end{aligned}
$$

which shows that C_q can be chosen appropriately. □

Remark 7.7 Note that the constants that realize the equivalences (7.3), (7.5), (7.8), in the sense of an upper and a lower bound, are in fact independent of α, since they only rely on the asymptotics of the $\Phi_{a,b}$ from Lemma 7.1. For instance, if we have

$$c_1\theta^{5+2\sigma} \le \Phi_{0,\sigma+1}(\theta) \le c_2\theta^{5+2\sigma}, \quad \theta \in [0,1],$$
$$c_3\theta^{4+\sigma} \le \Phi_{0,\sigma+1}(\theta) \le c_4\theta^{4+\sigma}, \quad \theta \in [1,\infty[,$$

for constants $c_2 > c_1 > 0$ and $c_4 > c_3 > 0$, then, following the proof of Lemma 7.3, it is not difficult to see that

$$c_1\alpha^{-3/2}\,(\alpha-\varepsilon)^{5/2+\sigma} \le G(\varepsilon) \le 2^{3/4}(1+\sqrt{2})^{5/2+\sigma}c_2\,\alpha^{-3/2}\,(\alpha-\varepsilon)^{5/2+\sigma}, \quad \varepsilon \in \Big[\frac{1}{\sqrt{2}}\,\alpha,\alpha\Big],$$
$$\Big(1-\frac{1}{\sqrt{2}}\Big)^{2+\sigma/2} c_3\,\varepsilon^{-3}\,\alpha^{4+\sigma} \le G(\varepsilon) \le \Big(1+\frac{1}{\sqrt{2}}\Big)^{2+\sigma/2} c_4\,\varepsilon^{-3}\,\alpha^{4+\sigma}, \quad \varepsilon \in \Big]0,\frac{1}{\sqrt{2}}\,\alpha\Big],$$

is obtained ♢

Let $\mathcal{G}(s) = \hat{G}(-s)$.

Lemma 7.8 (Properties of $\mathcal{G}$) *We have*

$$\mathcal{G}(0)=0, \quad \mathcal{G}'(0)=-\alpha, \quad \lim_{s\to\infty}\mathcal{G}'(s)=0, \quad \lim_{s\to\infty}\mathcal{G}(s)=-\infty,$$

and $\mathcal{G}(s) < 0$ *for* $s \in]0,\infty[$ *as well as* $\mathcal{G}'(s) \in]-\alpha,0[$ *for* $s \in]0,\infty[$. *Furthermore,* $\mathcal{G}''(s) > 0$ *for* $s \in]0,\infty[$ *and*

$$\mathcal{G}''(s) \sim \alpha^{\frac{3}{3+2\sigma}}\, s^{-\frac{1+2\sigma}{3+2\sigma}}, \quad s \to 0^+.$$

More precisely, let $c_1^*, c_2^*, c_3^*, c_4^*, c_5^* > 0$ *be constants (independent of* α*) such that*

$$-c_2^*\alpha^{-3/2}\,(\alpha-\varepsilon)^{3/2+\sigma} \le G'(\varepsilon) \le -c_1^*\alpha^{-3/2}\,(\alpha-\varepsilon)^{3/2+\sigma}, \quad \varepsilon \in \Big[\frac{1}{\sqrt{2}}\,\alpha,\alpha\Big], \tag{7.10}$$

and

$$G'(\varepsilon) \le -c_3^*\varepsilon^{-4}\alpha^{4+\sigma}, \quad \varepsilon \in \Big]0,\frac{1}{\sqrt{2}}\,\alpha\Big],$$

and moreover

$$c_4^*\alpha^{-3/2}\,(\alpha-\varepsilon)^{1/2+\sigma} \le G''(\varepsilon) \le c_5^*\alpha^{-3/2}\,(\alpha-\varepsilon)^{1/2+\sigma}, \quad \varepsilon \in \Big[\frac{1}{\sqrt{2}}\,\alpha,\alpha\Big], \tag{7.11}$$

are verified; recall (7.5) and (7.8). Put $s_0 = 2c_3^*\alpha^\sigma$. *Then*

$$(c_5^*)^{-1}(c_1^*)^{\frac{1+2\sigma}{3+2\sigma}}\,\alpha^{\frac{3}{3+2\sigma}}\,s^{-\frac{1+2\sigma}{3+2\sigma}} \le \mathcal{G}''(s) \le (c_4^*)^{-1}(c_2^*)^{\frac{1+2\sigma}{3+2\sigma}}\,\alpha^{\frac{3}{3+2\sigma}}\,s^{-\frac{1+2\sigma}{3+2\sigma}}, \tag{7.12}$$

for all $s \in]0,s_0]$.

Proof: See Section 7.2 for the notation. Since $\inf_{\varepsilon\in\mathbb{R}} G(\varepsilon) = 0$, we get $\mathcal{G}(0) = \hat{G}(0) = -\inf_{\varepsilon\in\mathbb{R}} G(\varepsilon) = 0$. For the derivative, if $u < 0$, then $\hat{G}(u)$ is attained at a unique $\varepsilon(u) \in]0, \alpha[$. As $\varepsilon(u) = (G')^{-1}(u)$ is the inverse function of G', we obtain $\lim_{u\to-\infty} \varepsilon(u) = 0$ and $\lim_{u\to 0^-} \varepsilon(u) = \alpha$ from Lemma 7.4. For $u \to 0^-$ we can hence use (7.3) and (7.5) to deduce

$$\frac{G(\varepsilon(u))}{u} = \frac{G(\varepsilon(u))}{G'(\varepsilon(u))} \sim \frac{\alpha^{-3/2}\,(\alpha-\varepsilon(u))^{5/2+\sigma}}{-\alpha^{-3/2}\,(\alpha-\varepsilon(u))^{3/2+\sigma}} = -(\alpha-\varepsilon(u)) \to 0,$$

so that also $\lim_{u\to 0^-} \frac{G(\varepsilon(u))}{u} = 0$. It follows that

$$\lim_{u\to 0^-} \frac{\hat{G}(0)-\hat{G}(u)}{-u} = \lim_{u\to 0^-} \frac{\hat{G}(u)}{u} = \lim_{u\to 0^-} \Big(\varepsilon(u) - \frac{G(\varepsilon(u))}{u}\Big) = \alpha,$$

which means that $\mathcal{G}'(0) = -\hat{G}'(0) = -\alpha$ does exist. Next, due to $\lim_{u\to-\infty} \hat{G}'(u) = \lim_{u\to-\infty} \varepsilon(u) = 0$, we also have $\lim_{s\to\infty} \mathcal{G}'(s) = 0$. If $s \in]0,\infty[$, then $u = -s \in]-\infty, 0[$ and therefore $\varepsilon(u) \in]0,\alpha[$. Since $\mathcal{G}''(s) = \frac{1}{G''(\varepsilon(u))}$ by (7.16), Lemma 7.5 implies that $\mathcal{G}''(s) > 0$. In particular, $\mathcal{G}'$ is increasing from $-\alpha$ to 0, which shows that $\mathcal{G}'(s) \in]-\alpha, 0[$ for $s \in]0,\infty[$. As a consequence of $\mathcal{G}(0) = 0$, this in turn yields $\mathcal{G}(s) < 0$ for $s \in]0,\infty[$. To verify $\lim_{s\to\infty} \mathcal{G}(s) = \lim_{u\to-\infty} \hat{G}(u) = -\infty$, we use (7.5) to deduce $G'(\varepsilon) \sim -\varepsilon^{-4}\alpha^{4+\sigma}$ as $\varepsilon \to 0^+$. Since $\varepsilon(u) \to 0^+$ as $u \to -\infty$, this gives $u = G'(\varepsilon(u)) \sim -\varepsilon(u)^{-4}\alpha^{4+\sigma}$ as $u \to -\infty$, and hence

$$\varepsilon(u)u \sim \Big(\frac{\alpha^{4+\sigma}}{-u}\Big)^{1/4} u = -\alpha^{1+\sigma/4}(-u)^{3/4}$$

as $u \to -\infty$. Also $\varepsilon(u) \to 0^+$ in conjunction with (7.3) shows that $G(\varepsilon(u)) \to \infty$ as $u \to -\infty$. Thus $\hat{G}(u) = \varepsilon(u)u - G(\varepsilon(u)) \to -\infty$ as $u \to -\infty$. To establish (7.12), fix $s \in]0, s_0]$ and let again $u = -s$. Suppose that $\varepsilon(u) \le \frac{1}{\sqrt{2}}\,\alpha$. Then $u = G'(\varepsilon(u)) \le -c_3^*\varepsilon(u)^{-4}\alpha^{4+\sigma}$, which implies that

$$\frac{1}{4}\,\alpha^4 \ge \varepsilon(u)^4 \ge c_3^*\alpha^{4+\sigma}\,\frac{1}{(-u)} \ge c_3^*\alpha^{4+\sigma}\,\frac{1}{s_0} = \frac{1}{2}\,\alpha^4,$$

which is a contradiction. Therefore we must have $\varepsilon(u) \in [\frac{1}{\sqrt{2}}\,\alpha, \alpha]$, and (7.10) applies. It follows that

$$-c_2^*\alpha^{-3/2}\,(\alpha-\varepsilon(u))^{3/2+\sigma} \le u \le -c_1^*\alpha^{-3/2}\,(\alpha-\varepsilon(u))^{3/2+\sigma},$$

and consequently

$$(c_2^*)^{-\frac{2}{3+2\sigma}}\,\alpha^{\frac{3}{3+2\sigma}}\,(-u)^{\frac{2}{3+2\sigma}} \le \alpha - \varepsilon(u) \le (c_1^*)^{-\frac{2}{3+2\sigma}}\,\alpha^{\frac{3}{3+2\sigma}}\,(-u)^{\frac{2}{3+2\sigma}}. \tag{7.13}$$

¿From (7.11) and $\mathcal{G}''(s) = \frac{1}{G''(\varepsilon(u))}$ we deduce that

$$(c_5^*)^{-1}\alpha^{3/2}\,(\alpha-\varepsilon(u))^{-(1/2+\sigma)} \le \mathcal{G}''(s) \le (c_4^*)^{-1}\alpha^{3/2}\,(\alpha-\varepsilon(u))^{-(1/2+\sigma)},$$

and hence we can use (7.13) to get

$$(c_5^*)^{-1}(c_1^*)^{\frac{1+2\sigma}{3+2\sigma}}\,\alpha^{3/2}\,\alpha^{-\frac{3(1+2\sigma)}{2(3+2\sigma)}}\,(-u)^{-\frac{1+2\sigma}{3+2\sigma}} \le \mathcal{G}''(s) \le (c_4^*)^{-1}(c_2^*)^{\frac{1+2\sigma}{3+2\sigma}}\,\alpha^{3/2}\,\alpha^{-\frac{3(1+2\sigma)}{2(3+2\sigma)}}\,(-u)^{-\frac{1+2\sigma}{3+2\sigma}},$$

as asserted. □

7.2 Legendre transforms

The Legendre transform of a general strictly convex real-valued function G is

$$\hat{G}(u) = \sup_{\varepsilon\in\mathbb{R}} (\varepsilon u - G(\varepsilon)) = -\inf_{\varepsilon\in\mathbb{R}} (G(\varepsilon) - \varepsilon u);$$

see [12, Section 8.1]. For every fixed u let $\varepsilon(u) \in \mathbb{R}$ be such that $\hat{G}(u) = \varepsilon(u)u - G(\varepsilon(u))$ and

$$0 = \frac{d}{d\varepsilon} [\varepsilon u - G(\varepsilon)]\Big|_{\varepsilon=\varepsilon(u)} = u - G'(\varepsilon(u)).$$

¿From the strict convexity of G it follows that G' is invertible. Therefore $\varepsilon(u) = (G')^{-1}(u)$, and then

$$\hat{G}'(u) = \varepsilon(u) + \varepsilon'(u)u - G'(\varepsilon(u))\varepsilon'(u) = \varepsilon(u)$$

shows that

$$\hat{G}'(u) = \varepsilon(u) = (G')^{-1}(u) \tag{7.14}$$

is the derivative of the Legendre transform.

The relation $\hat{\hat{G}} = G$ is also useful. To calculate the Legendre transform of $\hat{G}$, one first has to locate $u(\varepsilon)$ such that $0 = \varepsilon - \hat{G}'(u(\varepsilon))$, and then $\hat{\hat{G}}(\varepsilon) = u(\varepsilon)\varepsilon - \hat{G}(u(\varepsilon))$. Thus $u(\varepsilon) = G'(\varepsilon)$ is found, which leads to

$$\hat{\hat{G}}(\varepsilon) = G'(\varepsilon)\varepsilon - \hat{G}(G'(\varepsilon)) = G'(\varepsilon)\varepsilon - \varepsilon(G'(\varepsilon))G'(\varepsilon) + G(\varepsilon(G'(\varepsilon))) = G(\varepsilon)$$

as claimed, using that $\varepsilon(G'(\varepsilon)) = (G')^{-1}(G'(\varepsilon)) = \varepsilon$.

Next let $\mathcal{G}(s) = \hat{G}(-s)$. Then

$$\mathcal{G}(s) - s\,\mathcal{G}'(s) = -G(-\mathcal{G}'(s)). \tag{7.15}$$

To establish (7.15), we write $u = -s$ and get from $\mathcal{G}'(s) = -\hat{G}'(-s)$ that $\mathcal{G}(s) - s\,\mathcal{G}'(s) = \hat{G}(u) - u\,\hat{G}'(u) = \hat{G}(u) - \varepsilon(u)u = -G(\varepsilon(u))$, and thus the claim follows from $\varepsilon(u) = \hat{G}'(u) = \hat{G}'(-s) = -\mathcal{G}'(s)$.

Another noteworthy relation is

$$\mathcal{G}''(s) = \frac{1}{G''(\varepsilon(-s))}. \tag{7.16}$$

In fact, $\mathcal{G}''(s) = \hat{G}''(-s) = \hat{G}''(u)$, and (7.14) yields $G'(\hat{G}'(u)) = u$, which upon differentiation shows that $1 = G''(\hat{G}'(u))\hat{G}''(u) = G''(\varepsilon(u))\hat{G}''(u)$.

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