

# Existence of axially symmetric static solutions of the Einstein-Vlasov system

Håkan Andréasson  
Mathematical Sciences  
Chalmers University of Technology  
Göteborg University  
S-41296 Göteborg, Sweden  
email: hand@chalmers.se

Markus Kunze  
Fakultät für Mathematik  
Universität Duisburg-Essen  
D-45117 Essen, Germany  
email: markus.kunze@uni-due.de

Gerhard Rein  
Fakultät für Mathematik, Physik und Informatik  
Universität Bayreuth  
D-95440 Bayreuth, Germany  
email: gerhard.rein@uni-bayreuth.de

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## **Abstract**

We prove the existence of static, asymptotically flat non-vacuum spacetimes with axial symmetry where the matter is modeled as a collisionless gas. The axially symmetric solutions of the resulting Einstein-Vlasov system are obtained via the implicit function theorem by perturbing off a suitable spherically symmetric steady state of the Vlasov-Poisson system.

# 1 Introduction

The aim of the present investigation is to prove the existence of static, asymptotically flat, and axially symmetric solutions of the Einstein-Vlasov system. This system describes, in the context of general relativity, the evolution of an ensemble of particles which interact only via gravity. Examples from astrophysics of such ensembles include galaxies or globular clusters where the stars play the role of the particles and where collisions among these particles are usually sufficiently rare to be neglected. The particle distribution is given by a density function  $f$  on the tangent bundle  $TM$  of the spacetime manifold  $M$ . We assume that all particles have the same rest mass which is normalized to unity. Hence the particle distribution function is supported on the mass shell

$$PM = \{g_{\alpha\beta}p^\alpha p^\beta = -c^2 \text{ and } p^\alpha \text{ is future pointing}\} \subset TM.$$

Here  $g_{\alpha\beta}$  denotes the Lorentz metric on the spacetime  $M$  and if  $x^\alpha$  are coordinates on  $M$ , then  $p^\alpha$  denote the corresponding canonical momentum coordinates; Greek indices always run from 0 to 3, and we have a specific reason for making the dependence on the speed of light  $c$  explicit. We assume that the coordinates are chosen such that

$$ds^2 = c^2 g_{00} dt^2 + g_{ab} dx^a dx^b$$

where Latin indices run from 1 to 3 and  $t = x^0$  should be thought of as a timelike coordinate. On the mass shell  $p^0$  can be expressed by the remaining coordinates,

$$p^0 = \sqrt{-g^{00}} \sqrt{1 + c^{-2} g_{ab} p^a p^b},$$

and  $f = f(t, x^a, p^b) \geq 0$ . The Einstein-Vlasov system now consists of the Einstein field equations

$$G_{\alpha\beta} = 8\pi c^{-4} T_{\alpha\beta} \tag{1.1}$$

coupled to the Vlasov equation

$$p^0 \partial_t f + p^a \partial_{x^a} f - \Gamma_{\beta\gamma}^a p^\beta p^\gamma \partial_{p^a} f = 0 \tag{1.2}$$

via the following definition of the energy momentum tensor:

$$T_{\alpha\beta} = c|g|^{1/2} \int p_\alpha p_\beta f \frac{dp^1 dp^2 dp^3}{-p_0}. \tag{1.3}$$

Here  $|g|$  denotes the modulus of the determinant of the metric, and  $\Gamma_{\beta\gamma}^\alpha$  are the Christoffel symbols induced by the metric. We note that the characteristic system of the Vlasov equation (1.2) are the geodesic equations written

as a first order system on the mass shell  $PM$  which is invariant under the geodesic flow. For more background on the Einstein-Vlasov equation we refer to [3].

In [15, 17, 18] the existence of a broad variety of static, asymptotically flat solutions of this system has been established, all of which share the restriction that they are spherically symmetric. The purpose of the present investigation is to remove this restriction and prove the existence of static, asymptotically flat solutions to the Einstein-Vlasov system which are axially symmetric but not spherically symmetric. From the applications point of view this symmetry assumption is more “realistic” than spherical symmetry, and from the mathematics point of view the complexity of the Einstein field equations increases drastically if one gives up spherical symmetry.

We use usual axial coordinates  $t \in \mathbb{R}$ ,  $\rho \in [0, \infty[$ ,  $z \in \mathbb{R}$ ,  $\varphi \in [0, 2\pi]$  and write the metric in the form

$$ds^2 = -c^2 e^{2\nu/c^2} dt^2 + e^{2\mu} d\rho^2 + e^{2\mu} dz^2 + \rho^2 B^2 e^{-2\nu/c^2} d\varphi^2 \quad (1.4)$$

for functions  $\nu, B, \mu$  depending on  $\rho$  and  $z$ . The reason for writing  $\nu/c^2$  instead of  $\nu$  is so that below  $\nu$  converges to the Newtonian potential  $U_N$  in the limit  $c \rightarrow \infty$ . The metric is to be asymptotically flat in the sense that the boundary values

$$\lim_{|(\rho,z)| \rightarrow \infty} \nu(\rho, z) = \lim_{|(\rho,z)| \rightarrow \infty} \mu(\rho, z) = 0, \quad \lim_{|(\rho,z)| \rightarrow \infty} B(\rho, z) = 1 \quad (1.5)$$

are attained at spatial infinity with certain rates which are specified later. In addition we need to require the condition that the metric is locally flat at the axis of symmetry, i.e.,

$$\nu(0, z)/c^2 + \mu(0, z) = \ln B(0, z), \quad z \in \mathbb{R}. \quad (1.6)$$

We refer to [4] for more information on axially symmetric spacetimes and state our main result.

**Theorem 1.1** *There exist static solutions of the Einstein-Vlasov system (1.1), (1.2), (1.3) with  $c = 1$  such that the metric is of the form (1.4) and satisfies the boundary conditions (1.5), (1.6), and the spacetime is axially symmetric, but not spherically symmetric.*

It should be pointed out that the above form of the metric excludes solutions with non-zero total angular momentum. Since the corresponding generalization induces qualitatively new, additional difficulties it is postponed to a later investigation.

The strategy of the proof of this result is as follows. Due to the symmetries of the metric the following quantities are constant along geodesics:

$$\begin{aligned} E &:= -g(\partial/\partial t, p^\alpha) = c^2 e^{2\nu/c^2} p^0 \\ &= c^2 e^{\nu/c^2} \sqrt{1 + c^{-2} (e^{2\mu}(p^1)^2 + e^{2\mu}(p^2)^2 + \rho^2 B^2 e^{-2\nu/c^2} (p^3)^2)}, \end{aligned} \quad (1.7)$$

$$L := g(\partial/\partial\varphi, p^\alpha) = \rho^2 B^2 e^{-2\nu/c^2} p^3; \quad (1.8)$$

$E$  can be thought of as a local or particle energy and  $L$  is the angular momentum of a particle with respect to the axis of symmetry. Since up to regularity issues a distribution function  $f$  satisfies the Vlasov equation if and only if it is constant along geodesics, any distribution function  $f$  which depends only on  $E$  and  $L$  satisfies the Vlasov equation with a metric of the above form. Hence we make the ansatz

$$f(x^a, p^b) = \phi(E, L), \quad (1.9)$$

and the Vlasov equation (1.2) holds. Upon insertion of this ansatz into the definition (1.3) of the energy momentum tensor the latter becomes a functional  $T_{\alpha\beta} = T_{\alpha\beta}(\nu, B, \mu)$  of the yet unknown metric functions  $\nu, B, \mu$ , and we are left with the problem of solving the field Einstein equations (1.1) with this right hand side. We obtain solutions by perturbing off spherically symmetric steady states of the Vlasov-Poisson system via the implicit function theorem; the latter system arises as the Newtonian limit of the Einstein-Vlasov system. Our main result specifies conditions on the ansatz function  $\phi$  above such that a two parameter family of axially symmetric solutions of the Einstein-Vlasov system passes through the corresponding spherically symmetric, Newtonian steady state. The parameter  $\gamma = 1/c^2$  turns on general relativity and the second parameter  $\lambda$  turns on the dependence on  $L$  and hence axial symmetry; notice that  $L$  is not invariant under arbitrary rotations about the origin, so if  $f$  actually depends on  $L$  the solution is not spherically symmetric. The scaling symmetry of the Einstein-Vlasov system can then be used to obtain the desired solutions for the physically correct value of  $c$ .

The detailed formulation of our result is stated in the next section together with the basic set up of its proof. The remaining sections of the paper are then devoted to establishing the various features of the basic set up which are needed to apply the implicit function theorem, and to prove various properties of the solutions we obtain.

We conclude this introduction with some further references to the literature. The idea of using the implicit function theorem to obtain equilibrium

configurations of self-gravitating matter distributions from already known solutions can be traced back to L. LICHTENSTEIN who argued the existence of axially symmetric, stationary, self-gravitating fluid balls in this way [11, 12]. His arguments were put into a rigorous and modern framework in [8]. The analogous approach was used in [16] to obtain axially symmetric steady states of the Vlasov-Poisson system, see also [19]. The approach has also been used to construct axially symmetric stationary solutions of the Einstein equations coupled to a matter model: In [9] matter was described as an ideal fluid whereas in [1, 2] it was described as a static or a rotating elastic body respectively. Besides the different matter model our investigation differs from the latter two in that we employ the rather explicit form of the metric stated above and a reduced version of the Einstein field equations which closely follows [4].

## 2 Set up of the proof

In what follows we also use the Cartesian coordinates

$$(x^1, x^2, x^3) = (\rho \cos \varphi, \rho \sin \varphi, z) \in \mathbb{R}^3$$

which correspond to the axial coordinates  $\rho \in [0, \infty[$ ,  $z \in \mathbb{R}$ ,  $\varphi \in [0, 2\pi[$ ; it should be noted that tensor indices always refer to the spacetime coordinates  $t, \rho, z, \varphi$ . By abuse of notation we write  $\nu(\rho, z) = \nu(x)$  etc. In Section 3 we collect the relevant information on the relation between regularity properties of axially symmetric functions expressed in the variables  $x \in \mathbb{R}^3$  or  $\rho \in [0, \infty[$ ,  $z \in \mathbb{R}$ , respectively.

We introduce two (small) parameters  $\gamma = 1/c^2 \in [0, \infty[$  and  $\lambda \in \mathbb{R}$ . In order to obtain the correct Newtonian limit below we adjust the ansatz for  $f$  as follows. Let

$$v^1 = e^\mu p^1, \quad v^2 = e^\mu p^2, \quad v^3 = \rho B e^{-\gamma \nu} p^3,$$

so that

$$p^0 = e^{-\gamma \nu} \sqrt{1 + \gamma |v|^2}.$$

For the particle distribution function we make the ansatz

$$f(x, v) = \phi(E - 1/\gamma) \psi(\lambda L). \quad (2.1)$$

The important point here is that

$$E - 1/\gamma = \frac{e^{\gamma \nu(x)} \sqrt{1 + \gamma |v|^2} - 1}{\gamma} \rightarrow \frac{1}{2} |v|^2 + \nu(x) \text{ as } \gamma \rightarrow 0, \quad (2.2)$$

i.e., the limit is the non-relativistic energy of a particle with phase space coordinates  $(x, v)$  in case  $\nu = U_N$  is the Newtonian gravitational potential. For  $\gamma = 0$  this limit is to replace the argument of  $\phi$  in (2.1). We now specify the conditions on the functions  $\phi$  and  $\psi$ .

**Conditions on  $\phi$  and  $\psi$ .**

( $\phi 1$ )  $\phi \in C^2(\mathbb{R})$  and there exists  $E_0 > 0$  such that  $\phi(\eta) = 0$  for  $\eta \geq E_0$  and  $\phi(\eta) > 0$  for  $\eta < E_0$ .

( $\phi 2$ ) The ansatz  $f(x, v) = \phi\left(\frac{1}{2}|v|^2 + U(x)\right)$  leads to a compactly supported steady state of the Vlasov-Poisson system, i.e., there exists a solution  $U = U_N \in C^2(\mathbb{R}^3)$  of the semilinear Poisson equation

$$\Delta U = 4\pi\rho_N = 4\pi \int \phi\left(\frac{1}{2}|v|^2 + U\right) dv, \quad U(0) = 0,$$

$U_N(x) = U_N(|x|)$  is spherically symmetric, and the support of  $\rho_N \in C^2(\mathbb{R}^3)$  is the closed ball  $\overline{B}_{R_N}(0)$  where  $U_N(R_N) = E_0$  and  $U_N(r) < E_0$  for  $0 \leq r < R_N$ ,  $U_N(r) > E_0$  for  $r > R_N$ .

( $\phi 3$ )  $6 + 4\pi r^2 a_N(r) > 0, \quad r \in [0, \infty[$ ,

where

$$a_N(r) := \int_{\mathbb{R}^3} \phi'\left(\frac{1}{2}|v|^2 + U_N(r)\right) dv.$$

( $\psi$ )  $\psi \in C^\infty(\mathbb{R})$  is even with  $\psi(L) = 1$  iff  $L = 0$ , and  $\psi \geq 0$ .

For such a steady state

$$\lim_{|x| \rightarrow \infty} U_N(x) = U_N(\infty) > E_0.$$

The normalization condition  $U_N(0) = 0$  instead of  $U_N(\infty) = 0$  is unconventional from the physics point of view, but it has technical advantages below. Examples for ansatz functions  $\phi$  which satisfy ( $\phi 1$ ) and ( $\phi 2$ ) are found in [5, 18], the most well-known ones being the polytropes

$$\phi(E) := (E_0 - E)_+^k \tag{2.3}$$

for  $2 < k < 7/2$ ; here  $E_0 > 0$  and  $(\cdot)_+$  denotes the positive part. In Section 7 we show that for this class of ansatz functions also ( $\phi 3$ ) holds. Numerical checks indicate that ( $\phi 3$ ) holds for general isotropic steady states of the Vlasov-Poisson system.

We can now give a more detailed formulation of our result.

**Theorem 2.1** *There exists  $\delta > 0$  and a two parameter family*

$$(\nu_{\gamma,\lambda}, B_{\gamma,\lambda}, \mu_{\gamma,\lambda})_{(\gamma,\lambda) \in [0,\delta[\times]-\delta,\delta[ \subset C^2(\mathbb{R}^3)^3$$

*with the following properties:*

- (i)  $(\nu_{0,0}, B_{0,0}, \mu_{0,0}) = (U_N, 1, 0)$  where  $U_N$  is the potential of the Newtonian steady state specified in  $(\phi 2)$ .
- (ii) If for  $\gamma > 0$  a distribution function is defined by Eqn. (2.1) and a Lorentz metric by (1.4) with  $c = 1/\sqrt{\gamma}$  then this defines a solution of the Einstein-Vlasov system (1.1), (1.2), (1.3) which satisfies the boundary condition (1.6) and is asymptotically flat. For  $\lambda \neq 0$  this solution is not spherically symmetric.
- (iii) If for  $\gamma = 0$  a distribution function is defined by Eqn. (2.1), observing (2.2), this yields a steady state of the Vlasov-Poisson system with gravitational potential  $\nu_{0,\lambda}$  which is not spherically symmetric for  $\lambda \neq 0$ .
- (iv) In all cases the matter distribution is compactly supported both in phase space and in space.

**Remark.**

- (a) The smallness restriction to  $\gamma = 1/c^2$  is undesired because  $c$  is, in a given set of units, a definite number. However, if  $(f, \nu, B, \mu)$  is a static solution for some choice of  $c \in ]0, \infty[$  then the rescaling

$$\begin{aligned} \tilde{f}(\rho, z, p^1, p^2, p^3) &= c^{-3} f(c\rho, cz, cp^1, cp^2, p^3), \\ \tilde{\nu}(\rho, z) &= c^{-2} \nu(c\rho, cz), \\ \tilde{B}(\rho, z) &= B(c\rho, cz), \\ \tilde{\mu}(\rho, z) &= \mu(c\rho, cz) \end{aligned}$$

yields a solution of the Einstein-Vlasov system with  $c = 1$ . The factor  $c^2$  in the metric (1.4) is removed by a rescaling of time.

- (b) The smallness restriction to  $\lambda$  means that the solutions obtained are close to being spherically symmetric.
- (c) The metric does not satisfy the boundary conditions (1.5), but

$$\lim_{|(\rho,z)| \rightarrow \infty} \nu(\rho, z) = \nu_\infty, \quad \lim_{|(\rho,z)| \rightarrow \infty} \mu(\rho, z) = -\nu_\infty/c^2, \quad \lim_{|(\rho,z)| \rightarrow \infty} B(\rho, z) = 1. \quad (2.4)$$

However, if we by abuse of notation redefine  $\nu = \nu - \nu_\infty$  and  $\mu = \mu + \nu_\infty/c^2$  then the original condition (1.5) is restored and the metric (1.4) takes the form

$$ds^2 = -c^2 e^{2\nu/c^2} c_1^2 dt^2 + c_2^2 \left( e^{2\mu} d\rho^2 + e^{2\mu} dz^2 + \rho^2 B^2 e^{-2\nu/c^2} d\varphi^2 \right) \quad (2.5)$$

with constants  $c_1, c_2 > 0$  which simply amounts to a choice of different units of time and space. By general covariance of the Einstein-Vlasov system (1.1), (1.2), (1.3) the equations still hold.

- (d) In view of [16] part (iii) of the theorem does not give new information on steady states of the Vlasov-Poisson system and is stated mainly in order to understand the obtained two parameter family of states as a whole. However, we note that for the Newtonian set-up in [16] axially symmetric steady states were obtained as deformations of a spherically symmetric one. The present approach differs considerably from this and in principle is more direct.
- (e) In the course of the proof of the theorem additional regularity properties and specific rates at which the boundary values at infinity are approached will emerge.

In the rest of this section we transform the problem of finding the desired solutions into the problem of finding zeros of a suitably defined operator. The Newtonian steady state specified in ( $\phi 2$ ) will be a zero of this operator for  $\gamma = \lambda = 0$ , and the implicit function theorem will yield our result. In order that the overall course of the argument becomes clear we will go through its various steps, postponing the corresponding detailed proofs to later sections.

The Einstein field equations are overdetermined, and we need to identify a suitable subset of (combinations of) these equations which, on the one hand, suffice to determine  $\nu, B, \mu$ , and which are such that at the end of the day all the field equations hold once this reduced system is solved. We introduce the auxiliary metric function

$$\xi = \gamma\nu + \mu.$$

Let  $\Delta$  and  $\nabla$  denote the Cartesian Laplace and gradient operator respec-



tively. Taking suitable combinations of the field equations one finds that

$$\Delta\nu + \frac{\nabla B}{B} \cdot \nabla\nu = 4\pi\gamma \left[ \gamma e^{(2\xi - 4\gamma\nu)} T_{00} + T_{11} + T_{22} + \frac{1}{\rho^2 B^2} e^{2\xi} T_{33} \right], \quad (2.6)$$

$$\Delta B + \frac{\nabla\rho}{\rho} \cdot \nabla B = 8\pi\gamma^2 B (T_{11} + T_{22}), \quad (2.7)$$

$$\begin{aligned} \left(1 + \rho \frac{\partial_\rho B}{B}\right) \partial_\rho \xi - \rho \frac{\partial_z B}{B} \partial_z \xi \\ = \frac{1}{2\rho B} \partial_\rho (\rho^2 \partial_\rho B) - \frac{\rho}{2B} \partial_{zz} B + \gamma^2 \rho ((\partial_\rho \nu)^2 - (\partial_z \nu)^2), \end{aligned} \quad (2.8)$$

$$\left(1 + \rho \frac{\partial_\rho B}{B}\right) \partial_z \xi + \rho \frac{\partial_z B}{B} \partial_\rho \xi = \frac{\partial_\rho (\rho \partial_z B)}{B} + 2\gamma^2 \rho \partial_\rho \nu \partial_z \nu. \quad (2.9)$$

The last two equations arise from  $\rho(G_{11} - G_{22}) = 0$  and  $\rho G_{12} = 0$  respectively; note that due to (2.1),  $T_{11} = T_{22}$  and  $T_{12} = 0$ . Because of the asymptotic behavior of  $B$  and the structure of the left hand side of (2.7) we write

$$B = 1 + h/\rho.$$

Next, we observe that by taking suitable combinations of (2.8) and (2.9) we obtain equations which contain only  $\partial_\rho \xi$  or  $\partial_z \xi$  respectively, and we chose the former. In the above equations the terms  $T_{\alpha\beta}$  are functions of the unknown quantities  $\nu, h, \xi = \gamma\nu + \mu$  for which we therefore have obtained the following reduced system of equations:

$$\Delta\nu = 4\pi (\Phi_{00} + \gamma\Phi_{11} + \gamma\Phi_{33}) (\nu, B, \xi, \rho; \gamma, \lambda) - \frac{1}{B} \nabla(h/\rho) \cdot \nabla\nu, \quad (2.10)$$

$$\partial_{\rho\rho} h + \partial_{zz} h = 8\pi\gamma^2 \rho B \Phi_{11} (\nu, B, \xi, \rho; \gamma, \lambda), \quad (2.11)$$

$$\begin{aligned} ((1 + \partial_\rho h)^2 + (\partial_z h)^2) \partial_\rho \xi = \partial_z h (\partial_{z\rho} h + 2\gamma^2 (\rho + h) \partial_\rho \nu \partial_z \nu) \\ + (1 + \partial_\rho h) \left( \frac{1}{2} (\partial_{\rho\rho} h - \partial_{zz} h) + \gamma^2 (\rho + h) ((\partial_\rho \nu)^2 - (\partial_z \nu)^2) \right). \end{aligned} \quad (2.12)$$

We supplement this with the boundary condition (1.6) which in terms of the new unknowns and since necessarily  $h(0, z) = 0$ , reads

$$\xi(0, z) = \ln(1 + \partial_\rho h(0, z)). \quad (2.13)$$

It remains to determine precisely the dependence of the functions  $\Phi_{\alpha\beta}$  on the unknown quantities  $\nu, h, \xi$ . Since the ansatz (2.1) is even in the momentum

variables  $p^1, p^2, p^3$ —the fact that  $\psi$  is even is needed here—, all the off-diagonal elements of the energy-momentum tensor vanish. The computation of its non-trivial components uses the new integration variables

$$\eta = \frac{e^{\gamma\nu} \sqrt{1 + \gamma|v|^2} - 1}{\gamma}, \quad s = v^3,$$

the abbreviation

$$m(\eta, B, \nu, \gamma) = B e^{-\gamma\nu} \sqrt{\frac{e^{-2\gamma\nu} (1 + \gamma\eta)^2 - 1}{\gamma}},$$

and yields

$$\Phi_{00}(\nu, B, \xi, \rho; \gamma, \lambda) = \gamma^2 e^{(2\xi - 4\gamma\nu)} T_{00} \quad (2.14)$$

$$= \frac{4\pi}{B} e^{(2\xi - 4\gamma\nu)} \int_{(e^{\gamma\nu} - 1)/\gamma}^{\infty} \phi(\eta) (1 + \gamma\eta)^2 \int_0^{m(\eta, B, \nu, \gamma)} \psi(\lambda\rho s) ds d\eta,$$

$$\Phi_{11}(\nu, B, \xi, \rho; \gamma, \lambda) = T_{11} + T_{22} \quad (2.15)$$

$$= \frac{4\pi}{B^3} e^{2\xi} \int_{(e^{\gamma\nu} - 1)/\gamma}^{\infty} \phi(\eta) \int_0^{m(\eta, B, \nu, \gamma)} \psi(\lambda\rho s) (m^2(\eta, B, \nu, \gamma) - s^2) ds d\eta,$$

$$\Phi_{33}(\nu, B, \xi, \rho; \gamma, \lambda) = \frac{e^{2\xi}}{\rho^2 B^2} T_{33} \quad (2.16)$$

$$= \frac{4\pi}{B^3} e^{2\xi} \int_{(e^{\gamma\nu} - 1)/\gamma}^{\infty} \phi(\eta) \int_0^{m(\eta, B, \nu, \gamma)} \psi(\lambda\rho s) s^2 ds d\eta;$$

we recall that  $T_{11} = T_{22}$ . The reason for keeping  $B$  as argument on the right hand sides above is that the matter terms are differentiable in this variable, but taking a derivative with respect to  $h$  would yield an irritating factor  $1/\rho$ . For elements of the function space chosen below  $h/\rho$  extends smoothly to the axis of symmetry  $\rho = 0$ .

We now define the function spaces in which we will obtain the solutions of the system (2.10), (2.11), (2.12). As noted above we write, by abuse of notation, axially symmetric functions as functions of  $x \in \mathbb{R}^3$  or of  $\rho \geq 0, z \in \mathbb{R}$ ; regularity properties of axially symmetric functions are considered in Section 3. We fix  $0 < \alpha < 1/2$  and  $0 < \beta < 1$ , and consider the Banach spaces

$$\mathcal{X}_1 := \left\{ \nu \in C^{3,\alpha}(\mathbb{R}^3) \mid \nu(x) = \nu(\rho, z) = \nu(\rho, -z) \text{ and } \|\nu\|_{\mathcal{X}_1} < \infty \right\},$$

$$\mathcal{X}_2 := \left\{ h \in C^{4,\alpha}(\mathbb{R}^2) \mid h(\rho, z) = -h(-\rho, z) = h(\rho, -z) \text{ and } \|h\|_{\mathcal{X}_2} < \infty \right\},$$

$$\mathcal{X}_3 := \left\{ \xi \in C^{2,\alpha}(Z_R) \mid \xi(x) = \xi(\rho, z) = \xi(\rho, -z) \text{ and } \|\xi\|_{\mathcal{X}_3} < \infty \right\},$$

where

$$Z_R := \{x \in \mathbb{R}^3 \mid \rho < R\}$$

is the cylinder of radius  $R > 0$ , the latter being defined in (2.17) below. The norms are defined by

$$\begin{aligned} \|\nu\|_{\mathcal{X}_1} &:= \|\nu\|_{C^{3,\alpha}(\mathbb{R}^3)} + \|(1 + |x|)^{1+\beta} \nabla \nu\|_{\infty}, \\ \|h\|_{\mathcal{X}_2} &:= \|h\|_{C^{4,\alpha}(\mathbb{R}^2)} + \|(1 + |(\rho, z)|)^2 \nabla(h/\rho)\|_{\infty}, \\ \|\xi\|_{\mathcal{X}_3} &:= \|\xi\|_{C^{2,\alpha}(Z_R)}, \end{aligned}$$

and

$$\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3, \quad \|(\nu, h, \xi)\|_{\mathcal{X}} := \|\nu\|_{\mathcal{X}_1} + \|h\|_{\mathcal{X}_2} + \|\xi\|_{\mathcal{X}_3}.$$

Here  $\|\cdot\|_{\infty}$  denotes the  $L^{\infty}$ -norm, functions in  $C^{k,\alpha}(\mathbb{R}^n)$  have by definition continuous derivatives up to order  $k$  and all the highest order derivatives are Hölder continuous with exponent  $\alpha$ ,

$$\|g\|_{C^{k,\alpha}(\mathbb{R}^n)} := \sum_{|\sigma| \leq k} \|D^{\sigma} g\|_{\infty} + \sum_{|\sigma|=k} \sup_{x,y \in \mathbb{R}^n, x \neq y} \frac{|D^{\sigma} g(x) - D^{\sigma} g(y)|}{|x - y|^{\alpha}}.$$

and  $D^{\sigma}$  denotes the derivative corresponding to a multi-index  $\sigma \in \mathbb{N}_0^n$ . We note that if  $h \in \mathcal{X}_2$  then  $B = 1 + h/\rho \in C^3(\mathbb{R}^3)$ , cf. Lemma 3.2. Moreover, it will be straightforward to extend  $\xi$  to  $\mathbb{R}^3$  once a solution is obtained in the above space.

Now we recall the properties of the Newtonian steady state specified in  $(\phi 2)$ . That condition implies that there exists  $R > R_N > 0$  such that

$$U_N(r) > (E_0 + U_N(\infty))/2, \quad r > R. \quad (2.17)$$

If

$$\|\nu - U_N\|_{\infty} < |E_0 - U_N(\infty)|/4 \text{ and } 0 \leq \gamma < \gamma_0,$$

with  $\gamma_0 > 0$  sufficiently small, depending on  $E_0$  and  $U_N$ , then

$$\frac{e^{\gamma \nu(x)} - 1}{\gamma} > E_0 \text{ for all } |x| > R.$$

This implies that there exists some  $\delta > 0$  such that for all  $(\nu, h, \xi; \gamma, \lambda) \in \mathcal{U}$  the matter terms resulting from (2.14)–(2.16) are compactly supported in  $B_R(0)$ , where

$$\mathcal{U} := \{(\nu, h, \xi; \gamma, \lambda) \in \mathcal{X} \times [0, \delta[\times] - \delta, \delta[ \mid \|(\nu, h, \xi) - (U_N, 0, 0)\|_{\mathcal{X}} < \delta\}.$$

In addition we require that  $\delta > 0$  is sufficiently small so that for all elements in  $\mathcal{U}$  it holds that  $B = 1+h/\rho > 1/2$ , and the factor in front of  $\partial_\rho \xi$  in (2.12) is larger than  $1/2$ ; since  $h$  vanishes on the axis of symmetry,  $h/\rho$  is controlled by  $\nabla h$ . Now let an element  $(\nu, h, \xi; \gamma, \lambda) \in \mathcal{U}$  be given and substitute it into the matter terms defined in (2.14)–(2.16). With the right hand sides obtained in this way the equations (2.10)–(2.12) can then be solved, observing the boundary condition (2.13) and the fact that we require  $h$  to vanish on the axis of symmetry. We define the corresponding solution operators by

$$\begin{aligned} G_1(\nu, h, \xi; \gamma, \lambda)(x) &:= - \int_{\mathbb{R}^3} \left( \frac{1}{|x-y|} - \frac{1}{|y|} \right) M_1(y) dy \\ &\quad + \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\nabla(h/\rho)(y) \cdot \nabla \nu(y)}{B(y)} \frac{dy}{|x-y|}, \\ G_2(\nu, h, \xi; \gamma, \lambda)(x) &:= 4 \int_{\mathbb{R}^2} \ln |(\rho - \tilde{\rho}, z - \tilde{z})| \tilde{\rho} M_2(\tilde{\rho}, \tilde{z}) d\tilde{\rho} d\tilde{z}, \\ G_3(\nu, h, \xi; \gamma, \lambda)(x) &:= \ln(1 + \partial_\rho h(0, z)) + \int_0^\rho g(s, z) ds, \quad 0 \leq \rho < R. \end{aligned}$$

Here

$$\begin{aligned} M_1(x) &:= (\Phi_{00} + \gamma \Phi_{11} + \gamma \Phi_{33})(\nu(x), B(x), \xi(x), \rho; \gamma, \lambda), \\ M_2(\rho, z) &:= \gamma^2 B(x) \Phi_{11}(\nu(x), B(x), \xi(x), \rho; \gamma, \lambda), \end{aligned}$$

$M_2(\rho, z) = M_2(-\rho, z)$  for  $\rho < 0$  and  $z \in \mathbb{R}$ , and

$$\begin{aligned} g &:= ((1 + \partial_\rho h)^2 + (\partial_z h)^2)^{-1} \left[ \partial_z h (\partial_{z\rho} h + 2\gamma^2(\rho + h)\partial_\rho \nu \partial_z \nu) \right. \\ &\quad \left. + (1 + \partial_\rho h) \left( \frac{1}{2}(\partial_{\rho\rho} h - \partial_{zz} h) + \gamma^2(\rho + h) ((\partial_\rho \nu)^2 - (\partial_z \nu)^2) \right) \right]. \end{aligned} \quad (2.18)$$

Finally we define the mapping to which we are going to apply the implicit function theorem as

$$\mathcal{F} : \mathcal{U} \rightarrow \mathcal{X}, \quad (\nu, h, \xi; \gamma, \lambda) \mapsto (\nu, h, \xi) - (G_1, G_2, G_3)(\nu, h, \xi; \gamma, \lambda).$$

The proof of Theorem 2.1 now proceeds in a number of steps.

*Step 1.*

As a first step we need to check that the mapping  $\mathcal{F}$  is well defined, in particular it preserves the various regularity and decay assumptions. This is done in Section 4.

*Step 2.*

The next step is to see that

$$\mathcal{F}(U_N, 0, 0; 0, 0) = 0.$$

This is due to the fact that for  $\gamma = \lambda = 0$  the choice  $h = \xi = 0$  trivially satisfies (2.11), (2.12), while (2.10) reduces to

$$\Delta\nu = 4\pi\Phi_{00}(\nu, 1, 0; 0, 0)$$

with

$$\Phi_{00}(\nu, 1, 0; 0, 0) = 4\pi \int_{\nu}^{\infty} \phi(\eta) \sqrt{2(\eta - \nu)} d\eta = \int_{\mathbb{R}^3} \phi\left(\frac{1}{2}|v|^2 + \nu\right) dv;$$

notice that  $h = 0$  implies that  $B = 1$ . By ( $\phi 2$ ),  $\nu = U_N$  is a solution of this equation, and the fact that  $U_N \in \mathcal{X}_1$  is part of what was shown in the previous step.

*Step 3.*

Next we show that  $\mathcal{F}$  is continuous, and continuously Fréchet differentiable with respect to  $(\nu, h, \xi)$ . The fairly technical but straightforward details are covered in Section 5.

*Step 4.*

The crucial step is to see that the Fréchet derivative

$$L := D\mathcal{F}(U_N, 0, 0; 0, 0) : \mathcal{X} \rightarrow \mathcal{X}$$

is one-to-one and onto. Indeed,

$$L(\delta\nu, \delta h, \delta\xi) = (\delta\nu - L_1(\delta\nu) - L_2(\delta h), \delta h, \delta\xi - L_3(\delta h))$$

where

$$\begin{aligned} L_1(\delta\nu)(x) &:= - \int_{\mathbb{R}^3} \left( \frac{1}{|x-y|} - \frac{1}{|y|} \right) a_N(y) \delta\nu(y) dy, \\ L_2(\delta h)(x) &:= \frac{1}{4\pi} \int_{\mathbb{R}^3} \nabla(\delta h/\rho)(y) \cdot \nabla U_N(y) \frac{dy}{|x-y|}, \\ L_3(\delta h)(x) &:= \partial_\rho \delta h(0, z) + \frac{1}{2} \int_0^\rho (\partial_{\rho\rho} \delta h - \partial_{zz} \delta h)(s, z) ds, \quad 0 \leq \rho < R, \end{aligned}$$

with  $a_N$  as defined in ( $\phi 3$ ). To see that  $L$  is one-to-one let  $L(\delta\nu, \delta h, \delta\xi) = 0$ . Then the second component of this identity implies that  $\delta h = 0$ , and hence

also  $\delta\xi = 0$  by the third component. It therefore remains to show that  $\delta\nu = 0$  is the only solution of the equation  $\delta\nu = L_1(\delta\nu)$ , i.e., of the equation

$$\Delta\delta\nu = 4\pi a_N \delta\nu, \quad \delta\nu(0) = 0 \quad (2.19)$$

in the space  $\mathcal{X}_1$ . Under the assumption on  $a_N$  stated in  $(\phi 3)$  this is correct and shown in Section 6. It is at this point that our unconventional normalization condition in  $(\phi 2)$  together with the shift in the solution operator  $G_1$  become important; notice that  $L_1(\delta\nu)(0) = 0$ .

To see that  $L$  is onto let  $(g_1, g_2, g_3) \in \mathcal{X}$  be given. We need to show that there exists  $(\delta\nu, \delta h, \delta\xi) \in \mathcal{X}$  such that  $L(\delta\nu, \delta h, \delta\xi) = (g_1, g_2, g_3)$ . The second component of this equation simply says that  $\delta h = g_2$ . Now  $\delta h \in \mathcal{X}_2$  implies that  $L_3(\delta h) \in \mathcal{X}_3$ , cf. Lemma 3.1 (b). Hence we set  $\delta\xi = g_3 + L_3(\delta h)$  to satisfy the third component of the onto equation, and it remains to show that the equation

$$\delta\nu - L_1(\delta\nu) = g_1 + L_2(\delta h) \quad (2.20)$$

has a solution  $\delta\nu \in \mathcal{X}_1$ . Firstly,  $L_2(\delta h) \in \mathcal{X}_1$ . The assertion therefore follows from the fact that  $L_1 : \mathcal{X}_1 \rightarrow \mathcal{X}_1$  is compact, as is shown in Lemma 6.2.

We are now ready to apply the implicit function theorem, cf. [7, Thm. 15.1], to the mapping  $\mathcal{F} : \mathcal{U} \rightarrow \mathcal{X}$ ; strictly speaking we should suitably extend  $\mathcal{F}$  to  $\gamma < 0$ , but this is not essential. We obtain the following result.

**Theorem 2.2** *There exists  $\delta_1, \delta_2 \in ]0, \delta[$  and a unique, continuous solution map*

$$S : [0, \delta_1[ \times ] - \delta_1, \delta_1[ \rightarrow B_{\delta_2}(U_N, 0, 0) \subset \mathcal{X}$$

such that  $S(0, 0) = (U_N, 0, 0)$  and

$$\mathcal{F}(S(\gamma, \lambda); \gamma, \lambda) = 0 \text{ for all } (\gamma, \lambda) \in [0, \delta_1[ \times ] - \delta_1, \delta_1[.$$

The definition of  $\mathcal{F}$  implies that for any  $(\gamma, \lambda)$  the functions  $(\nu, h, \xi) = S(\gamma, \lambda)$  are a solution of the equations (2.10)–(2.12), and if  $f$  is defined by (2.1) then the equations (2.6), (2.7), (2.12) hold with the induced energy momentum tensor. We can extend  $\xi$  to the whole space using the solution operator  $G_3$  for all  $x \in \mathbb{R}^3$ . Also, the boundary condition (1.6) on the axis of symmetry is satisfied:

$$\xi(0, z) = G_3(\nu, h, \xi)(0, z) = \ln(1 + \partial_\rho h(0, z)) = \ln B(0, z);$$

recall that  $\xi = \gamma\nu + \mu$ . For  $\gamma = 0$  we conclude first that  $h = 0$ , cf. (2.11) or the  $G_2$ -part of the solution operator respectively, then the  $G_3$ -part implies that  $\xi = 0$  so that the solution reduces to  $(\nu, 0, 0)$  where  $\nu$  solves

$$\Delta\nu = 4\pi\Phi_{00}(\nu, 1, 0, \rho; 0, \lambda).$$

Since

$$\Phi_{00}(\nu, 1, 0, \rho; 0, \lambda) = 4\pi \int_{\nu}^{\infty} \phi(\eta) \int_0^{\sqrt{2(\eta-\nu)}} \psi(\lambda \rho s) ds d\eta$$

coincides with the spatial density induced by the ansatz (2.1) for the Newtonian case, cf. [16, Lemma 2.1], part (iii) of Theorem 2.1 is established. If  $\lambda \neq 0$  then condition  $(\psi)$  implies that  $f$  really depends on the angular momentum variable  $L$  which is not invariant under all rotations about the origin, but only invariant under rotations about the axis  $\rho = 0$ . Moreover, if the metric were spherically symmetric then the explicit dependence of the quantities  $\Phi_{jj}$  on  $\rho$  would imply that the induced energy momentum tensor would not be spherically symmetric which is a contradiction. Hence the obtained solutions are not spherically symmetric if  $\lambda \neq 0$ . To complete the proof of Theorem 2.1 we must show that indeed *all* the field equations are satisfied by the obtained metric (1.4). The corresponding argument relies on the Bianchi identity  $\nabla_{\alpha} G^{\alpha\beta} = 0$  which holds for the Einstein tensor induced by any (sufficiently regular) metric, and on the identity  $\nabla_{\alpha} T^{\alpha\beta} = 0$  which is a direct consequence of the Vlasov equation (1.2);  $\nabla_{\alpha}$  denotes the covariant derivative corresponding to the metric (1.4). The details are carried out in Section 8.

Finally we collect the additional information on the solution which we obtain in the course of the proof.

**Proposition 2.3** *Let  $(\nu, h, \xi) = S(\gamma, \lambda)$  be any of the solutions obtained in Theorem 2.2 and define  $\mu := \xi - \nu/c^2$  and  $B = 1 + h/\rho$ . Then the limit  $\nu_{\infty} := \lim_{|x| \rightarrow \infty} \nu(x)$  exists, and for any  $\sigma \in \mathbb{N}_0^3$  with  $|\sigma| \leq 1$  and  $x \in \mathbb{R}^3$  the following estimates hold:*

$$\begin{aligned} |D^{\sigma}(\nu(x) - \nu_{\infty})| &\leq C(1 + |x|)^{-(1+|\sigma|)}, \\ |D^{\sigma}(B - 1)(x)| &\leq C(1 + |x|)^{-(2+|\sigma|)}, \\ |D^{\sigma}\xi(x)| &\leq C(1 + |x|)^{-(2+|\sigma|)}. \end{aligned}$$

*In particular, the spacetime equipped with the metric (1.4) is asymptotically flat in the sense that (2.4) and, after a trivial change of coordinates, also (1.5) holds.*

**Proof.** By definition of  $G_1$ ,  $\lim_{|x| \rightarrow \infty} \nu(x) = \int \frac{M_1(y)}{|y|} dy$ . The first two estimates are established in Lemma 4.2. As to the third one we observe that by the boundary condition (2.13) and Lemma 4.2,

$$|\xi(0, z)| \leq C |\partial_{\rho} h(0, z)| \leq \frac{C}{(1 + |z|)^2}.$$

By (2.12) and the known asymptotic behavior of the coefficients in that equation which are given in terms of  $\nu$  and  $h$  and their derivatives,

$$|\partial_\rho \xi(\rho, z)| \leq \frac{C}{1 + \rho^3 + |z|^3},$$

cf. Lemma 4.2. Hence

$$\begin{aligned} |\xi(\rho, z)| &\leq |\xi(0, z)| + \int_0^\rho |\partial_\rho \xi(s, z)| ds \\ &\leq \frac{C}{1 + |z|^2} + C \int_0^\infty \frac{ds}{1 + s^3 + |z|^3} \leq \frac{C}{1 + |z|^2} \end{aligned}$$

which is the desired estimate for  $\xi(\rho, z)$ , provided  $\rho < |z|$ . Since we already know that the metric under consideration satisfies the full set of the Einstein equations we can now use (2.8) and (2.9) to see that also  $\partial_z \xi$  is given in terms of  $\nu$  and  $h$  and their derivatives and satisfies the same decay estimate as  $\partial_\rho \xi$ . Starting from

$$|\xi(\rho, z)| \leq |\xi(\rho, \rho)| + \int_z^\rho |\partial_z \xi(\rho, s)| ds,$$

we can use the decay of  $\partial_z \xi$  to obtain the decay estimate for  $\xi(\rho, z)$  for  $\rho \geq z \geq 0$  (or  $\rho \geq -z \geq 0$ ), and the proof is complete.  $\square$

### 3 Regularity of axially symmetric functions

We call a function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  *axially symmetric* if there exists a function  $\tilde{f} : [0, \infty[ \times \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(x) = \tilde{f}(\rho, z), \text{ where } \rho = \sqrt{x_1^2 + x_2^2} \text{ and } z = x_3 \text{ for } x \in \mathbb{R}^3.$$

In this section we collect some results on the relation between the regularity properties of  $f$  and those of  $\tilde{f}$ .

**Lemma 3.1** *Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  be axially symmetric and  $f(x) = \tilde{f}(\rho, z)$  where  $\tilde{f} : [0, \infty[ \times \mathbb{R} \rightarrow \mathbb{R}$ . Let  $k \in \{1, 2, 3\}$  and  $\alpha \in ]0, 1[$ .*

- (a)  *$f \in C^k(\mathbb{R}^3)$  iff  $\tilde{f} \in C^k([0, \infty[ \times \mathbb{R})$  and all derivatives of  $\tilde{f}$  of order up to  $k$  which are of odd order in  $\rho$  vanish for  $\rho = 0$ .*
- (b)  *$f$  is Hölder continuous with exponent  $\alpha \in ]0, 1[$  iff  $\tilde{f}$  is.*



**Proof.** As to part (a) let  $f \in C^k(\mathbb{R}^3)$  be axially symmetric. Then  $f$  is even in  $x_1$  and  $x_2$  and  $\tilde{f}(\rho, z) = f(\rho, 0, z)$ . This proves the “only-if” part. For the “if” part one checks that the corresponding derivatives of  $f$ , which exist for  $\rho \neq 0$ , extend continuously to  $\rho = 0$ . As to part (b) one only needs to observe that  $x \mapsto \rho(x) = \sqrt{x_1^2 + x_2^2}$  is Lipschitz, since  $|\nabla \rho(x)| = 1$ .  $\square$

At several places in our analysis it is convenient to extend functions of  $(\rho, z)$  to negative values of  $\rho$ .

**Lemma 3.2** *Let  $h = h(\rho, z) \in C^4(\mathbb{R}^2)$  be odd in  $\rho$  and define*

$$b(\rho, z) := \begin{cases} h(\rho, z)/\rho, & \rho \neq 0, \\ \partial_\rho h(0, z), & \rho = 0. \end{cases}$$

*Then  $b \in C^3(\mathbb{R}^2)$  and all derivatives of  $b$  up to order 3 which are of odd order in  $\rho$  vanish for  $\rho = 0$ . By abuse of notation,  $b \in C^3(\mathbb{R}^3)$ .*

**Proof.** The regularity of  $b$  only needs to be checked at  $\rho = 0$ . Since  $h$  is odd in  $\rho$  it follows that  $h(0, z) = \partial_\rho h(0, z) = 0$  for  $z \in \mathbb{R}$ . Hence as  $\rho \rightarrow 0$ ,

$$b(\rho, z) = \frac{1}{\rho}(h(\rho, z) - h(0, z)) \rightarrow \partial_\rho h(0, z),$$

and by Taylor expansion,

$$\begin{aligned} \partial_\rho b(\rho, z) &= \frac{1}{\rho} \partial_\rho h(\rho, z) - \frac{1}{\rho^2} h(\rho, z) \\ &= \frac{1}{\rho} (\partial_\rho h(0, z) + \partial_{\rho\rho} h(\tau, z)\rho) \\ &\quad - \frac{1}{\rho^2} \left( h(0, z) + \partial_\rho h(0, z)\rho + \frac{1}{2} \partial_{\rho\rho} h(\sigma, z)\rho^2 \right) \\ &= \partial_{\rho\rho} h(\tau, z) - \frac{1}{2} \partial_{\rho\rho} h(\sigma, z) \\ &\rightarrow \frac{1}{2} \partial_{\rho\rho} h(0, z) = 0 \end{aligned} \tag{3.1}$$

where  $\sigma, \tau$  are between 0 and  $\rho$ . All other derivatives can be treated in a similar fashion, where one should observe that  $\partial_z h(0, z) = 0$ . The regularity with respect to  $x$  then follows by Lemma 3.1.  $\square$

## 4 $\mathcal{F}$ is well defined

As a first step we investigate the regularity properties of the functions  $\Phi_{jj}$ ,  $j = 0, \dots, 3$ , and of the induced matter terms  $M_1, M_2$ .

**Lemma 4.1** *Let  $\phi$  and  $\psi$  satisfy the conditions  $(\phi 1)$  and  $(\psi)$  respectively.*

- (a) *The functions  $\Phi_{00}$  and  $\Phi_{33}$  have derivatives with respect to  $\nu, \xi, \rho \in \mathbb{R}$  and  $B \in ]1/2, 3/2[$  up to order three and these are continuous in  $\nu, \xi, B, \rho, \gamma, \lambda$ . The same is true for  $\Phi_{11}$  for derivatives up to order four.*
- (b) *For  $(\nu, \xi, h; \gamma, \lambda) \in \mathcal{U}$ ,  $M_1 \in C^2(\mathbb{R}^3)$  and  $M_2 \in C^{2,\alpha}(\mathbb{R}^2)$  are both compactly supported.*

**Proof.** As to part (a) we note that differentiability with respect to  $\xi$  and  $\rho$  is straight forward. Concerning differentiability with respect to  $\nu$  and  $B$  we observe that for  $j = 0, \dots, 3$  the expression  $\Phi_{jj}$  is differentiable once with respect to the indicated variables, provided  $\phi \in L_{\text{loc}}^\infty$ , cf. the proof of [16, Lemma 2.1]. Under the assumption  $(\phi 1)$  we can first differentiate twice before the change to the integration variables  $\eta$  and  $s$  and obtain expressions which are essentially of the same form as  $\Phi_{jj}$ , but with  $\phi'$  or  $\phi''$  instead of  $\phi$  so that the resulting expression can be differentiated once more. The reason why  $\Phi_{11}$  is one order more differentiable is that when differentiating this expression with respect to  $\nu$  or  $B$  the integral with respect to  $s$  is preserved, its integrand is differentiated, and the resulting expression is qualitatively of the same type as  $\Phi_{00}$  and can be differentiated three more times.

Part (b) follows since the functions  $\nu, B, \xi$  which are now substituted into  $\Phi_{jj}$  are all at least in  $C^{2,\alpha}(\mathbb{R}^3)$ ; the fact that  $\xi$  is defined only on the cylinder  $Z_R$  does not matter here because the integrals in the definitions of  $\Phi_{jj}$  yield functions with support in  $Z_R$ .  $\square$

We now show that  $\mathcal{F}$  is well defined, more precisely:

**Lemma 4.2** *Let  $(\nu, \xi, h; \gamma, \lambda) \in \mathcal{U}$ . Then the following holds.*

- (a)  *$G_1 = G_1(\nu, \xi, h; \gamma, \lambda) \in C^{3,\alpha}(\mathbb{R}^3)$  is axially symmetric, even in  $z = x_3$ , and  $\|(1+|x|)(G_1 - a)\|_\infty, \|(1+|x|)^2 \nabla G_1\|_\infty < \infty$ , where  $a = \int \frac{M_1(y)}{|y|} dy$ .*

- (b)  *$G_2 = G_2(\nu, \xi, h; \gamma, \lambda) \in C^{4,\alpha}(\mathbb{R}^2)$  is odd in  $\rho$ , even in  $z$ , and*

$$\|(1+|(\rho, z)|)G_2\|_\infty, \|(1+|(\rho, z)|)^2 D^1 G_2\|_\infty, \|(1+|(\rho, z)|)^3 D^2 G_2\|_\infty < \infty.$$

Moreover,

$$\|(1+|(\rho, z)|)^2 (G_2/\rho)\|_\infty, \|(1+|(\rho, z)|)^3 D^1 (G_2/\rho)\|_\infty < \infty.$$

Here  $D^j$  stands for any derivative of order  $j$  with respect to  $(\rho, z) \in \mathbb{R}^2$ .

(c)  $G_3 = G_3(\nu, \xi, h; \gamma, \lambda) \in C^{2,\alpha}(Z_R)$  is axially symmetric, even in  $z = x_3$ , and  $\|G_3\|_{C^{2,\alpha}(Z_R)} < \infty$ .

(d)  $\mathcal{F}(\nu, \xi, h; \gamma, \lambda) \in \mathcal{X}$ .

**Proof.** As to part (a) the potential induced by the matter term  $M_1$ , which is in  $C^{1,\alpha}(\mathbb{R}^3)$  by Lemma 4.1 (b), has the desired regularity and decay properties due to standard regularity results in Hölder spaces, cf. [13, Thms. 10.2, 10.3], and the decay of  $1/|x - y|$  and its derivatives together with the compact support of  $M_1$ . As to the source term  $g = \nabla(h/\rho) \cdot \nabla\nu$  of the second term in  $G_1$  we notice that  $\nu \in \mathcal{X}_1$  and  $h \in \mathcal{X}_2$  implies that  $g \in C^{1,\alpha}(\mathbb{R}^3)$  with  $|g(x)| \leq C(1 + |x|)^{-3-\beta}$ , in particular  $g \in L^1 \cap L^\infty(\mathbb{R}^3)$ . This implies the regularity of the potential induces by  $g$  and also its decay:

$$\begin{aligned} \int \frac{|g(y)|}{|x - y|} dy &\leq \int_{|x-y| \leq |x|/2} \dots + \int_{|x-y| > |x|/2} \\ &\leq C \int_{|x-y| \leq |x|/2} (1 + |y|)^{-3-\beta} \frac{dy}{|x - y|} + \frac{2}{|x|} \int |g(y)| dy \\ &\leq C(1 + |x|/2)^{-3-\beta} \int_{|x-y| \leq |x|/2} \frac{dy}{|x - y|} + \frac{C}{|x|} \leq \frac{C}{|x|} \end{aligned}$$

for large  $|x|$  as desired; for the gradient of the potential induced by  $g$  we argue completely analogously.

As to part (b) we first recall that  $M_2 = M_2(\rho, z)$  is even in  $\rho$ , and the actual source term  $\rho M_2$  is odd, compactly supported, and by Lemma 4.1 (b) and Lemma 3.1 (b),  $M_2 \in C^{2,\alpha}(\mathbb{R}^2)$ . Hence  $G_2 \in C^{4,\alpha}(\mathbb{R}^2)$  is odd in  $\rho \in \mathbb{R}$ . As to the decay of  $G_2$  let  $\text{supp } M_2 \subset B_R(0) \subset \mathbb{R}^2$ . Then for  $|(\rho, z)| \geq 2R$  and  $(\tilde{\rho}, \tilde{z}) \in \text{supp } M_2$  the estimate

$$|\ln |(\rho - \tilde{\rho}, z - \tilde{z})| - \ln |(\rho, z)|| \leq \frac{2R}{|(\rho, z)|}$$

holds, and since  $\int \tilde{\rho} M_2 = 0$  this implies that

$$|G_2(\rho, z)| = \left| G_2(\rho, z) - 4 \int \ln |(\rho, z)| \tilde{\rho} M_2(\tilde{\rho}, \tilde{z}) d\tilde{z} d\tilde{\rho} \right| \leq \frac{C}{|(\rho, z)|};$$

the estimates for the derivatives of  $G_2$  follow along the same lines. Finally,  $\partial_\rho(G_2/\rho) = -G_2/\rho^2 + \partial_\rho G_2/\rho$  which implies that

$$|\partial_\rho(G_2/\rho)(\rho, z)| \leq \frac{C}{|(\rho, z)|\rho^2} + \frac{C}{|(\rho, z)|^2|\rho|}.$$

This yields the asserted decay when  $|\rho|$  becomes large. But we can also use (3.1) to see that  $|\partial_\rho(G_2/\rho)(\rho, z)| \leq C/|z|^3$ . Both estimates together yield the asserted decay for  $\partial_\rho(G_2/\rho)$ , and the decay for  $G_2/\rho$  and  $\partial_z(G_2/\rho)$  can be dealt with similarly.

In order to prove part (c) we observe that (2.18) and the regularity of  $\nu$  and  $h$  imply that  $g$  and hence  $G_3 \in C^{2,\alpha}(Z_R)$ . By construction,  $\partial_\rho G_3 = g$ . Since  $h$  is odd in  $\rho$  we find that

$$h(0, z) = \partial_z h(0, z) = \partial_{zz} h(0, z) = \partial_{\rho\rho} h(0, z) = 0,$$

which implies that  $g(0, z) = 0$ . Thus by Lemma 3.1,  $G_3 \in C^{2,\alpha}(Z_R)$ , and the proof is complete.  $\square$

## 5 $\mathcal{F}$ is continuous and continuously differentiable with respect to $\nu, h, \xi$

In this section we give some details of the proof of the following result:

**Lemma 5.1** *The mappings*

$$G_i : \mathcal{U} \rightarrow \mathcal{X}_i, \quad i = 1, 2, 3$$

*are continuous and continuously Fréchet differentiable with respect to  $\nu, h$ , and  $\xi$ .*

**Proof.** We only show the differentiability assertion and focus on  $G_1$ . Defining  $\Phi = \Phi_{00} + \gamma\Phi_{11} + \gamma\Phi_{33}$  we consider the differentiability only with respect to  $\nu$ , and neglecting the dependence on the remaining variables we look at the prototype mapping

$$G : \mathcal{V} \rightarrow \mathcal{X}_1, \quad G(\nu)(x) := \int_{\mathbb{R}^3} \frac{\Phi(\nu(y))}{|x-y|} dy,$$

where  $\mathcal{V} \subset \mathcal{X}_1$  is open,  $\Phi \in C^3(\mathbb{R})$  and  $\Phi \circ \nu$  has support in a fixed ball for all  $\nu \in \mathcal{V}$ . Our first claim is that  $G$  has the Fréchet derivative

$$[DG(\nu)\delta\nu](x) = \int_{\mathbb{R}^3} \frac{\Phi'(\nu(y))\delta\nu(y)}{|x-y|} dy, \quad \nu \in \mathcal{V}, \quad \delta\nu \in \mathcal{X}_1.$$

In order to prove this claim we need to show that for  $\nu \in \mathcal{V}$  there exists  $\epsilon > 0$  such that for  $\delta\nu \in B_\epsilon(0) \subset \mathcal{X}_1$ ,

$$\|G(\nu + \delta\nu) - G(\nu) - DG(\nu)\delta\nu\|_{\mathcal{X}_1} = o(\|\delta\nu\|_{\mathcal{X}_1}).$$

The support property and the standard elliptic estimate imply that

$$\begin{aligned}
& \|G(\nu + \delta\nu) - G(\nu) - DG(\nu)\delta\nu\|_{\mathcal{X}_1} \\
& \leq C \|G(\nu + \delta\nu) - G(\nu) - DG(\nu)\delta\nu\|_{C^{3,\alpha}(\mathbb{R}^3)} \\
& \leq C \|\Phi(\nu + \delta\nu) - \Phi(\nu) - \Phi'(\nu)\delta\nu\|_{C^{1,\alpha}(\mathbb{R}^3)} \\
& \leq C \|\Phi(\nu + \delta\nu) - \Phi(\nu) - \Phi'(\nu)\delta\nu\|_{C_b^2(\mathbb{R}^3)}.
\end{aligned}$$

Clearly,

$$\|\Phi(\nu + \delta\nu) - \Phi(\nu) - \Phi'(\nu)\delta\nu\|_\infty = o(\|\delta\nu\|_\infty) \leq o(\|\delta\nu\|_{\mathcal{X}_1}).$$

We need to establish analogous estimates for expressions where we take derivatives with respect to  $x$  up to second order of the left hand side. Let  $i, j \in \{1, 2, 3\}$ . Then

$$\begin{aligned}
\partial_{x_i} (\Phi(\nu + \delta\nu) - \Phi(\nu) - \Phi'(\nu)\delta\nu) &= (\Phi'(\nu + \delta\nu) - \Phi'(\nu)) \partial_{x_i} \delta\nu \\
&\quad + (\Phi'(\nu + \delta\nu) - \Phi'(\nu) - \Phi''(\nu)\delta\nu) \partial_{x_i} \nu
\end{aligned}$$

where both terms on the right are  $o(\|\delta\nu\|_{\mathcal{X}_1})$ . Similarly,

$$\begin{aligned}
& \partial_{x_i x_j} (\Phi(\nu + \delta\nu) - \Phi(\nu) - \Phi'(\nu)\delta\nu) \\
&= (\Phi''(\nu + \delta\nu) - \Phi''(\nu) - \Phi'''(\nu)\delta\nu) \partial_{x_i} \nu \partial_{x_j} \nu \\
&\quad + (\Phi''(\nu + \delta\nu) - \Phi''(\nu)) (\partial_{x_i} \nu \partial_{x_j} \delta\nu + \partial_{x_j} \nu \partial_{x_i} \delta\nu) \\
&\quad + (\Phi'(\nu + \delta\nu) - \Phi'(\nu) - \Phi''(\nu)\delta\nu) \partial_{x_i x_j} \nu \\
&\quad + (\Phi'(\nu + \delta\nu) - \Phi'(\nu)) \partial_{x_i x_j} \delta\nu + \Phi''(\nu + \delta\nu) \partial_{x_i} \delta\nu \partial_{x_j} \nu,
\end{aligned}$$

and all the terms appearing on the right are  $o(\|\delta\nu\|_{\mathcal{X}_1})$ . This proves the differentiability assertion for  $G$ . As to the continuity of this derivative,

$$\begin{aligned}
\|DG(\nu) - DG(\tilde{\nu})\|_{L(\mathcal{X}_1, \mathcal{X}_1)} &= \sup_{\|\delta\nu\|_{\mathcal{X}_1} \leq 1} \left\| \int_{\mathbb{R}^3} \frac{[(\Phi'(\nu) - \Phi'(\tilde{\nu}))\delta\nu](y)}{|\cdot - y|} dy \right\|_{\mathcal{X}_1} \\
&\leq C \sup_{\|\delta\nu\|_{\mathcal{X}_1} \leq 1} \|(\Phi'(\nu) - \Phi'(\tilde{\nu}))\delta\nu\|_{C^{1,\alpha}(\mathbb{R}^3)} \\
&\leq C \|\Phi'(\nu) - \Phi'(\tilde{\nu})\|_{C_b^2(\mathbb{R}^3)} \rightarrow 0 \text{ as } \tilde{\nu} \rightarrow \nu \text{ in } \mathcal{X}_1.
\end{aligned}$$

These arguments prove the continuous Fréchet differentiability of the first part of  $G_1$  with respect to  $\nu$ . The derivatives with respect to  $h$  or  $\xi$  can be dealt with in exactly the same manner. The source term in the potential which represents the second part of  $G_1$  can be expanded explicitly in powers of  $\delta h$  and  $\delta\nu$  which together with the standard elliptic estimate proves the

assertion for that term; note that both  $B$  and  $B + \delta h/\rho$  are bounded away from 0.

The mapping  $G_2$  is treated in the same way as our prototype  $G$  above, except that we have to estimate the source term including its third order derivatives, observing that  $\Phi_{11}$  has derivatives up to order four with respect to  $\nu, B, \xi$ .

The mapping  $G_3$  is easier since the term  $g$  defined in (2.18) can be expanded explicitly in powers of  $\delta\nu$  and  $\delta h$  where again we observe that the denominator in that expression is bounded away from 0.  $\square$

## 6 $D\mathcal{F}(U_N, 0, 0; 0, 0)$ is one-to-one and onto

We recall from Section 2 and Eqn. (2.19) that in order to prove that the map  $L$  is one-to-one it remains to show that  $g = 0$  is the only solution of

$$\Delta g = 4\pi a_N g, \quad g(0) = 0, \quad (6.1)$$

in the space  $\mathcal{X}_1$ . Inspired by the method in [16] we expand  $g$  into spherical harmonics  $Y_{lm}$ ,  $l \in \mathbb{N}_0$ ,  $m = -l, \dots, l$ , where we use the notation of [10]; for a more mathematical reference on spherical harmonics see [14]. Denote by  $(r, \theta, \varphi)$  and  $(s, \tau, \psi)$  the spherical coordinates of a point  $x \in \mathbb{R}^3$  and  $y \in \mathbb{R}^3$  respectively. For  $l \in \mathbb{N}_0$  and  $m = -l, \dots, l$  we define

$$g_{lm}(r) := \frac{1}{r^2} \int_{|x|=r} Y_{lm}^*(\theta, \varphi) g(x) dS_x. \quad (6.2)$$

The symmetry assumptions in the function space  $\mathcal{X}_1$  imply that  $g_{1-1} = g_{10} = g_{11} = 0$ , since up to multiplicative constants the spherical harmonics with  $l = 1$  are given by  $\sin \theta e^{\pm i\varphi}$  and  $\cos \theta$ . To proceed, we use the following expansion, cf. [10],

$$\frac{1}{|x - y|} = \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}^*(\tau, \psi) Y_{lm}(\theta, \varphi),$$

where  $r_< := \min(r, s)$  and  $r_> := \max(r, s)$ . In view of (6.1),

$$\begin{aligned}
g_{lm}(r) &= -\frac{1}{r^2} \int_{\mathbb{R}^3} \int_{|x|=r} \frac{1}{|x-y|} Y_{lm}^*(\theta, \varphi) dS_x a_N(s) g(y) dy \\
&= -\frac{4\pi}{2l+1} \int_0^\infty a_N(s) \frac{r_<^l}{r_>^{l+1}} \int_{|y|=s} Y_{lm}^*(\tau, \psi) g(y) dS_y ds \\
&= -\frac{4\pi}{2l+1} \int_0^\infty a_N(s) \frac{r_<^l}{r_>^{l+1}} s^2 g_{lm}(s) ds \\
&= -\frac{4\pi}{2l+1} \left( \int_0^r a_N(s) \frac{s^{l+2}}{r^{l+1}} g_{lm}(s) ds + \int_r^\infty a_N(s) \frac{r^l}{s^{l-1}} g_{lm}(s) ds \right).
\end{aligned}$$

By a straightforward computation we find that  $g_{lm}$  satisfies the equation

$$(r^2 g'_{lm})' = (l(l+1) + 4\pi r^2 a_N(r)) g_{lm}, \quad (6.3)$$

where prime denotes a derivative with respect to  $r$ .

We use this to show that  $g_{00} = 0$  as follows. We define  $w(r) := \sup_{0 \leq s \leq r} |g'_{00}(s)|$  so that  $|g_{00}(r)| \leq r w(r)$ ; at this point it becomes essential that  $g(0) = g_{00}(0) = 0$ . Now (6.3) can be integrated to yield the Gronwall estimate

$$w(r) \leq 4\pi \int_0^r s |a_N(s)| w(s) ds, \quad r \geq 0,$$

so that  $w = 0$  and hence  $g_{00} = 0$  as desired.

It therefore remains to consider  $g_{lm}$  with  $l \geq 2$ . For these we prove the following auxiliary result.

**Lemma 6.1** *Let  $a \in C_c([0, \infty[)$  and  $\lambda > 0$  be such that  $\lambda + 4\pi r^2 a(r) > 0$  for  $r \in [0, \infty[$ . Let  $u \in C^2([0, \infty[)$  be a bounded solution to*

$$(r^2 u')' = (\lambda + 4\pi r^2 a(r)) u. \quad (6.4)$$

*Then  $u = 0$ .*

**Proof.** We fix  $r_a > 0$  such that  $a(r) = 0$  for  $r \geq r_a$ . Multiplying (6.4) with  $u$  and integrating by parts we obtain for  $r > 0$ ,

$$\begin{aligned}
\int_0^r (\lambda + 4\pi s^2 a(s)) u^2(s) ds &= \int_0^r (s^2 u'(s))' u(s) ds \\
&= r^2 u'(r) u(r) - \int_0^r s^2 (u'(s))^2 ds. \quad (6.5)
\end{aligned}$$

Now if there exists  $r_0 > 0$  so that  $u(r_0) = 0$  or  $u'(r_0) = 0$  then (6.5) implies that  $u(r) = u'(r) = 0$  for  $r \in [0, r_0]$ . The unique solvability of (6.4) for  $r \geq r_0$  then shows that  $u = 0$  as claimed.

So we assume now that  $u(r) \neq 0$  and  $u'(r) \neq 0$  for  $r > 0$ . Since (6.4) is invariant under  $u \rightarrow -u$ , we may suppose that  $u(r) > 0$  and  $u'(r) > 0$  for all  $r \in ]0, \infty[$ ; note that (6.5) enforces  $uu' > 0$  on  $]0, \infty[$ . For  $r \geq r_a > 0$  (6.4) simplifies to  $(r^2 u')' = \lambda u$ , which has the solution

$$u(r) = \frac{(l+1)u(r_a) + r_a u'(r_a)}{(2l+1)} \left(\frac{r}{r_a}\right)^l + \frac{l u(r_a) - r_a u'(r_a)}{(2l+1)} \left(\frac{r_a}{r}\right)^{l+1}.$$

Therefore  $u$  is unbounded which is a contradiction.  $\square$

Since  $g \in \mathcal{X}_1$ , Eqn. (6.2) implies that  $g_{lm}$  is bounded. Due to  $(\phi 3)$  we can apply Lemma 6.1 to conclude that  $g_{lm} = 0$  for all  $l \geq 2$ , and thus  $g = 0$  as desired.

We now prove the compactness result which was needed to show that  $L$  is onto.

**Lemma 6.2** *The mapping  $K : \mathcal{X}_1 \rightarrow \mathcal{X}_1$ ,*

$$(Kw)(x) = \int_{\mathbb{R}^3} \frac{a_N(y) w(y)}{|x-y|} dy$$

*is compact.*

We remark that the operator  $L_1$  has the form  $L_1(\delta\nu)(x) = -K(\delta\nu)(x) + K(\delta\nu)(0)$  and is compact if  $K$  is, since the mapping  $\nu \mapsto \nu(0)$  is continuous on  $\mathcal{X}_1$ .

**Proof.** First we observe that the mapping

$$u \mapsto \int_{\mathbb{R}^3} \phi' \left( \frac{1}{2}|v|^2 + u \right) dv = 2\sqrt{2}\pi \int_u^\infty \phi'(E) \sqrt{E-u} dE$$

is in  $C^2(\mathbb{R})$ , and since  $U_N \in C^2(\mathbb{R}^3)$  the function  $a_N$  is in  $C_c^2(\mathbb{R}^3)$ . Hence  $a_N w \in C^{1,1/2}(\mathbb{R}^3)$  for any  $w \in \mathcal{X}_1$ , and since  $\alpha < 1/2$  the mapping  $K$  is well defined.

We fix a function  $\chi \in C_c^\infty(\mathbb{R}^3)$  such that  $0 \leq \chi \leq 1$ ,  $\chi(x) = 1$  for  $|x| \leq 1$ , and  $\chi(x) = 0$  for  $|x| \geq 2$ . Let  $\chi_R(x) = \chi(x/R)$  for  $R > 0$  and define

$$(K_R w)(x) = \chi_R(x)(Kw)(x).$$

We show that  $K_R \rightarrow K$  in the operator norm as  $R \rightarrow \infty$ . To this end, let  $\zeta_R = 1 - \chi_R$  so that for  $w \in \mathcal{X}_1$  and  $x \in \mathbb{R}^3$ ,

$$(Kw - K_R w)(x) = \zeta_R(x)(Kw)(x), \tag{6.6}$$



and the latter vanishes for  $|x| \leq R$ . Now let  $\|w\|_{\mathcal{X}_1} \leq 1$ . For  $\sigma \in \mathbb{N}_0^3$  with  $|\sigma| \leq 3$  it follows that

$$\begin{aligned} |D^\sigma(Kw - K_R w)(x)| &\leq \zeta_R(x) |D^\sigma(Kw)(x)| \\ &\quad + \sum_{0 < \tau \leq \sigma} |c_\tau D^\tau \zeta_R(x) D^{\sigma-\tau}(Kw)(x)| \\ &\leq \mathbf{1}_{\{|x| \geq R\}} \frac{C}{|x|} + C \sum_{0 < \tau \leq \sigma} \frac{1}{R} |D^{\sigma-\tau}(Kw)(x)| \leq \frac{C}{R}; \end{aligned}$$

constants denoted by  $C$  do not depend on  $x$  or  $R$ . In order to estimate the Hölder norm of  $D^\sigma(Kw - K_R w)$  for  $|\sigma| = 3$  we take  $x, \tilde{x} \in \mathbb{R}^3$  with  $|\tilde{x}| \geq |x|$  and again apply the product rule to the expression (6.6). Adding and subtracting terms we have to estimate expressions like

$$|(D^\tau \zeta_R(x) - D^\tau \zeta_R(\tilde{x})) D^{\sigma-\tau}(Kw)(x)| \leq \frac{C}{R} |x - \tilde{x}|$$

and terms like the following:

$$|D^\tau \zeta_R(x)| \int_{\mathbb{R}^3} |D^{\sigma-\tau}|y|^{-1}| |(a_N w)(x-y) - (a_N w)(\tilde{x}-y)| dy;$$

if  $|\sigma - \tau| = 3$  we throw one derivative onto  $a_N w$ . The latter quantity together with its first order derivatives is Hölder continuous. The factor in front of the integral vanishes for  $|x| \leq R$ , so we need only consider  $|\tilde{x}| \geq |x| \geq R$ . Since the domain of integration extends only over  $y$  with  $|y - x| \leq R_N$  or  $|y - \tilde{x}| \leq R_N$  we can on the domain of integration estimate  $|y| \geq |x| - |x - y| \geq R - R_N \geq R/2$  or analogously with  $\tilde{x}$  instead of  $x$ , where we assume that  $R > 2R_N$ . Since  $|D^{\sigma-\tau}|y|^{-1}| \leq |y|^{-j}$  with  $j \geq 1$  the term under consideration can be estimated by  $CR^{-1}|x - \tilde{x}|^\alpha$  and altogether we conclude that

$$\|Kw - K_R w\|_{C^{3,\alpha}(\mathbb{R}^3)} \leq C/R.$$

Recalling the definition of the norm  $\|\cdot\|_{\mathcal{X}_1}$  we see that the following chain of estimates finally shows that  $K_R \rightarrow K$  in the corresponding operator norm as desired:

$$\begin{aligned} |\nabla(Kw)(x) - \nabla(K_R w)(x)| &\leq \zeta_R(x) \int_{\mathbb{R}^3} \frac{|a_N(y)w(y)|}{|x-y|^2} dy \\ &\quad + R^{-1} |\nabla \chi(x/R)| \int_{\mathbb{R}^3} \frac{|a_N(y)w(y)|}{|x-y|} dy \\ &\leq \mathbf{1}_{\{|x| \geq R\}} \frac{C}{|x|^2} + R^{-1} \mathbf{1}_{\{R \leq |x| \leq 2R\}} \frac{C}{|x|} \\ &\leq C(1 + |x|)^{-\beta-1} R^{-(1-\beta)}. \end{aligned}$$

To complete the proof we have to show that  $K_R$  is compact for any  $R > 0$  on the space  $\mathcal{X}_1$ . First the fact that  $a_N \in C_c^2(\mathbb{R}^3)$  implies that

$$K_R : C^{3,\alpha}(\mathbb{R}^3) \rightarrow C^{3,1/2}(\mathbb{R}^3)$$

is continuous, and the same is true for

$$K_R : C^{3,\alpha}(\mathbb{R}^3) \rightarrow C^{3,1/2}(\overline{B_{3R}(0)})$$

where we note that all the functions  $K_R w$  with  $w \in C^{3,\alpha}(\mathbb{R}^3)$  are supported in  $B_{3R}(0)$ . Since  $\alpha < 1/2$  the embedding

$$C^{3,1/2}(\overline{B_{3R}(0)}) \hookrightarrow C^{3,\alpha}(\overline{B_{3R}(0)})$$

is compact, and because of the support property we conclude that

$$K_R : \mathcal{X}_1 \rightarrow \mathcal{X}_1$$

is compact; on  $\nabla K_R w$  the weight  $(1+|x|)^{1+\beta}$  only amounts to multiplication with a bounded function.  $\square$

## 7 Discussion of Condition ( $\phi 3$ )

In this section we investigate Condition ( $\phi 3$ ) for the case of the polytropic steady states (2.3). We first allow for the general range  $k \in ]-1/2, 7/2[$  of polytropic exponent. Using the elementary integration formula

$$\int_{\mathbb{R}^3} \left( s - \frac{1}{2}|v|^2 \right)_+^k dv = (2\pi)^{3/2} \frac{\Gamma(k+1)}{\Gamma(k+\frac{5}{2})} s_+^{k+\frac{3}{2}}, \quad s \in \mathbb{R}, \quad (7.1)$$

the Poisson equation in ( $\phi 2$ ) is found to be

$$\frac{1}{r^2} (r^2 U_N')' = 4\pi (2\pi)^{3/2} \frac{\Gamma(k+1)}{\Gamma(k+\frac{5}{2})} (E_0 - U_N)_+^{k+\frac{3}{2}}$$

for  $U_N = U_N(r)$ . According to [18] there exists a solution  $U_N$  such that  $U_N(0) < E_0$ ,  $U_N'(0) = 0$ ,  $U_N(R_N) = E_0$ ,  $U_N(r) > E_0$  for  $r > R_N$ , and  $U_N'(r) > 0$  for  $r \in ]0, R_N[$ . For  $z := E_0 - U_N$  this means that

$$-\frac{1}{r^2} (r^2 z')' = 4\pi c_n z_+^n, \quad \text{where } n := k + \frac{3}{2} \in ]1, 5[, \quad c_n := (2\pi)^{3/2} \frac{\Gamma(k+1)}{\Gamma(k+\frac{5}{2})},$$

and furthermore  $z(0) > 0$ ,  $z'(0) = 0$ ,  $z(R_N) = 0$ , and  $z'(r) < 0$  for  $r \in ]0, R_N[$ . In terms of  $z$  the function  $a_N$  from  $(\phi 3)$  reads

$$a_N(r) = -(2\pi)^{3/2} \frac{k\Gamma(k)}{\Gamma(k + \frac{3}{2})} z(r)_+^{k+\frac{1}{2}} = -n c_n z(r)_+^{n-1},$$

where once more (7.1) was used. Thus condition  $(\phi 3)$  is equivalent to

$$4\pi n c_n r^2 z(r)_+^{n-1} < 6. \quad (7.2)$$

Now consider the function  $\zeta(s) := z(\alpha s)$  for  $\alpha := (4\pi c_n)^{-1/2}$ . It is found to satisfy the Emden-Fowler equation

$$-\frac{1}{s^2}(s^2\zeta')' = \zeta_+^n \quad (7.3)$$

and  $\zeta(0) > 0$ ,  $\zeta'(0) = 0$ ,  $\zeta(s_0) = 0$  for  $s_0 := R_N/\alpha$ , as well as  $\zeta'(s) < 0$  for  $s \in ]0, s_0[$ . In terms of  $s = \alpha^{-1}r$  condition (7.2) becomes

$$s^2\zeta(s)_+^{n-1} < \frac{6}{n}. \quad (7.4)$$

The left-hand side can be conveniently expressed by means of the dynamical systems representation of (7.3). For, let

$$U(t) := -\frac{s\zeta(s)^n}{\zeta'(s)} \geq 0, \quad V(t) := -\frac{s\zeta'(s)}{\zeta(s)} \geq 0, \quad t := \ln s,$$

where we consider  $t \in ]-\infty, \ln s_0[$ . Then

$$\dot{U} = U(3 - U - nV), \quad \dot{V} = V(U + V - 1), \quad (7.5)$$

and  $U(t)V(t) = s^2\zeta(s)_+^{n-1}$ , which provides the relation to (7.4). Thus we have to verify that  $U(t)V(t) < 6/n$ . In the terminology of [6, p. 501], where  $m = 0$ ,  $\zeta$  is an  $E$ -solution to (7.3). Thus [6, Prop. 5.5] implies that  $(U(t), V(t))$  lies in the unstable manifold of the fixed point  $P_3 = (3, 0)$  of (7.5). In particular, we have  $\lim_{t \rightarrow -\infty} (U(t), V(t)) = (3, 0)$ . Also note that  $P_3$  is of saddle type with eigenvalues  $-3$  and  $2$ ; the corresponding eigenvectors are  $(1, 0)$  and  $(-3n/5, 1)$ . Since the line  $V = \frac{1}{n}(3 - U)$  separates the regions  $\dot{U} > 0$  (below the line) and  $\dot{U} < 0$  (above the line), a phase plane analysis reveals that we must always have  $U(t) \leq 3$ , so that  $W(t) := U(t)V(t) \leq 3V(t)$ . In addition, it is calculated that  $V$  and  $W$  are solutions to the system

$$\dot{V} = V(V - 1) + W, \quad \dot{W} = W(2 - (n - 1)V), \quad (7.6)$$

such that  $\lim_{t \rightarrow -\infty} (V(t), W(t)) = (0, 0)$ . The origin is a fixed point of saddle type for (7.6), the eigenvalues are  $-1$  and  $2$  with corresponding eigenvectors  $(1, 0)$  and  $(1, 3)$ . Note that  $\dot{W} > 0$  for  $V < \frac{2}{n-1}$ ,  $\dot{W} < 0$  for  $V > \frac{2}{n-1}$ ,  $\dot{V} > 0$  above the curve  $V \mapsto V(1 - V)$ , and  $\dot{V} < 0$  below this curve. Since the curve has unity slope at  $V = 0$ , it follows that  $(V(t), W(t))$ , lying in the unstable manifold of the origin, will be above the curve for  $t$  very negative. Then a phase plane analysis shows that this property persists for all times. In particular, we always have  $\dot{V} > 0$ , and  $W$  is increasing until it reaches its maximal value for  $t_0$  such that  $V(t_0) = \frac{2}{n-1}$ . Thus our original problem of proving  $(\phi 3)$  is equivalent to showing that  $W(t_0) = \max W < 6/n$ . Thanks to the preceding observations the parametrized curve  $t \mapsto (V(t), W(t))$  for  $t \in ]-\infty, t_0]$  can be rewritten as a curve  $W = W(V)$  in the  $(V, W)$ -plane which solves

$$\frac{dW}{dV} = \frac{W(2 - (n-1)V)}{V(V-1) + W}, \quad (7.7)$$

and which is such that  $W(0) = 0$  and  $W(\frac{2}{n-1}) = \max W$ .

**Lemma 7.1** *If  $k < 7/2$  is sufficiently close to  $7/2$ , then  $(\phi 3)$  holds for  $\phi$  given by (2.3).*

**Proof.** If  $W(v) < 1 < 6/n$  for all  $V \in ]0, \frac{2}{n-1}]$ , then we are done. Hence we assume that  $W(V_0) = 1$  for some  $V_0 \in ]0, \frac{2}{n-1}]$ . Then  $1 = W(V_0) \leq 3V_0$  yields  $V_0 \geq 1/3$ . Since  $W(V) \geq 1$  for  $V \geq V_0$ , it follows that  $V(V-1) + W = (V-1)^2 + V + W - 1 \geq V$ , so that by (7.7),

$$\begin{aligned} \ln(\max W) &= \int_{W=1}^{\max W} \frac{dW}{W} \leq \int_{V_0}^{\frac{2}{n-1}} \frac{(2 - (n-1)\tilde{V})}{\tilde{V}} d\tilde{V} \\ &\leq \int_{1/3}^{\frac{2}{n-1}} \frac{(2 - (n-1)\tilde{V})}{\tilde{V}} d\tilde{V} \\ &= 2 \ln \left( \frac{6}{n-1} \right) - 2 + \frac{1}{3}(n-1). \end{aligned}$$

Therefore

$$\max W \leq \frac{36}{(n-1)^2} \exp \left( \frac{1}{3}(n-1) - 2 \right). \quad (7.8)$$

At  $n = 5$  the relation

$$\frac{9}{4} e^{-2/3} < \frac{6}{5}$$

holds. Hence it follows from (7.8) that  $\max W < 6/n$  is verified for  $n$  sufficiently close to  $n = 5$ .  $\square$

The method of proof for the preceding lemma can be refined as follows. Fix  $A < 6/n$ . Then  $W(V) < A$  for  $V \in [0, \frac{2}{n-1}]$  would be acceptable. Hence we can assume that  $W(V_0) = A$  for some  $V_0 \in ]0, \frac{2}{n-1}]$ . Then  $A = W(V_0) \leq 3V_0$  shows that  $V_0 \geq A/3$ . From  $W(V) \geq A$  for  $V \geq V_0$  we obtain

$$\begin{aligned} & \ln(\max W) - \ln A \\ &= \int_{W(V_0)}^{\max W} \frac{dW}{W} \leq \int_{V_0}^{\frac{2}{n-1}} \frac{(2 - (n-1)\tilde{V})}{\tilde{V}^2 - \tilde{V} + A} d\tilde{V} \\ &\leq \int_{A/3}^{\frac{2}{n-1}} \frac{(2 - (n-1)\tilde{V})}{\tilde{V}^2 - \tilde{V} + A} d\tilde{V} \\ &= \frac{5-n}{\sqrt{4A-1}} \left[ \arctan\left(\frac{5-n}{\sqrt{4A-1}(n-1)}\right) + \arctan\left(\frac{3-2A}{3\sqrt{4A-1}}\right) \right] \\ &\quad - \left(\frac{n-1}{2}\right) \ln\left(\frac{9[An^2 - 2(A+1)n + 6 + A]}{A(A+6)(n-1)^2}\right). \end{aligned}$$

Therefore  $\max W \leq \Phi_A(n)$  where

$$\begin{aligned} \Phi_A(n) &:= A \left( \frac{A(A+6)(n-1)^2}{9[An^2 - 2(A+1)n + 6 + A]} \right)^{\frac{n-1}{2}} \\ &\quad \times \exp\left( \frac{5-n}{\sqrt{4A-1}} \left[ \arctan\left(\frac{5-n}{\sqrt{4A-1}(n-1)}\right) + \arctan\left(\frac{3-2A}{3\sqrt{4A-1}}\right) \right] \right). \end{aligned}$$

For different  $A$  it can be checked (e.g. using Maple) for which values  $n \in ]1, \min\{6/A, 5\}[$  the relation  $\Phi_A(n) < 6/n$  is verified. Taking  $A = 1$  we get at least  $n \in [2.6, 5[$ , for  $A = 6/5$  we get at least  $n \in [2.35, 4.85]$ , and for  $A = 2$  we get at least  $n \in [2.1, 2.5]$ . In summary, the desired relation  $\max W < 6/n$  can be obtained for at least  $n \in [2.1, 5[$ , which corresponds to at least  $k \in [0.6, 3.5[$  in (2.3). Notice however that the regularity assumption on  $\phi$  requires  $k > 2$ .

## 8 The field equations hold

For a metric of the form (1.4) the components 00, 11, 12, 22, and 33 of the field equations are nontrivial. We have so far obtained a solution  $\nu, B, \xi$  of the reduced system (2.6), (2.7), (2.12) where the appearing components of the energy momentum tensor are induced by a phase space density  $f$  which satisfies the Vlasov equation (1.2). We define  $E_{\alpha\beta} := G_{\alpha\beta} - 8\pi c^{-4}T_{\alpha\beta}$  so that the Einstein field equations become  $E_{\alpha\beta} = 0$ . By (2.7),

$$E_{11} + E_{22} = 0. \tag{8.1}$$

Using this information (2.6) says that

$$\rho^2 B^2 E_{00} + c^2 e^{4\nu/c^2} E_{33} = 0$$

or

$$c^2 e^{4\nu/c^2} E^{00} + \rho^2 B^2 E^{33} = 0. \quad (8.2)$$

The Vlasov equation implies that  $\nabla_\alpha T^{\alpha\beta} = 0$ , and  $\nabla_\alpha G^{\alpha\beta} = 0$  due to the contracted Bianchi identity where  $\nabla_\alpha$  denotes the covariant derivative corresponding to the metric (1.4). We want to use these relations to show that the remaining components of  $E_{\alpha\beta}$  vanish also, but there is a technical catch: The metric, more specifically  $\xi$ , is only  $C^2$ . To overcome this complication we approximate  $\xi$  by  $C^3$  functions  $\xi_n$ . The induced Einstein tensor  $G_n^{\alpha\beta}$  again satisfies the Bianchi identity. Taking  $\beta = 1$  and letting  $n \rightarrow \infty$  we obtain the equation

$$\partial_z E^{12} + \left(4\partial_z \mu + \frac{\partial_z B}{B}\right) E^{12} - \rho B e^{-2\xi} (B + \rho \partial_\rho B) E^{33} = 0, \quad (8.3)$$

where (8.2) has been used to eliminate  $E^{00}$  and we recall that  $\xi = \nu/c^2 + \mu$ . Here  $\partial_z E^{12}$  is at first a distributional derivative, but since all other terms in the equation are continuous this derivative indeed exists in the classical sense. The same approximation maneuver can be performed for  $\beta = 2$  to obtain the equation

$$\partial_\rho E^{12} + \left(4\partial_\rho \mu + \frac{1}{\rho} + \frac{\partial_\rho B}{B}\right) E^{12} - \rho^2 B e^{-2\xi} \partial_z B E^{33} = 0 \quad (8.4)$$

which holds for  $\rho > 0$ . However, if we multiply this equation with  $\rho$  we obtain an equation which holds for  $\rho \geq 0$ . This is because  $E^{12}(0, z) = 0$  which is nothing but the boundary condition (2.13) on the axis of symmetry which we have incorporated into our integration of (2.12). We eliminate  $E^{33}$  from (8.3), (8.4) and write the resulting equation for  $E^{12}$  in terms of

$$X := \rho e^{4\mu} B E^{12}.$$

The result is the equation

$$\partial_\rho X - \frac{\rho \partial_z B}{B + \rho \partial_\rho B} \partial_z X = 0$$

which again holds for  $\rho \geq 0$ . Since  $X(0, z) = 0$  and since any characteristic curve of this equation intersects the axis of symmetry  $\rho = 0$  we conclude

that  $X$  vanishes identically. By (8.3) the same is true for  $E^{33}$  so that  $E^{12} = E^{33} = E^{00} = 0$ . Finally we observe that by (2.12),

$$\left(1 + \rho \frac{\partial \rho B}{B}\right) (E_{11} - E_{22}) + \rho \frac{\partial_z B}{B} E_{12} = 0.$$

Since  $E_{12} = 0$  this means that  $E_{11} = E_{22}$ , and with (8.1) we conclude that  $E_{11} = E_{22} = 0$ , and all the non-trivial field equations are satisfied.

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