Multipole Radiation in Classical Particle Systems

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1. Models of Individual Particles

The motion of N charged classical particles in $\mathbb{R}^3$ under the influence of their self-generated electromagnetic fields can be described by the Abraham-Lorentz system

$$\frac{d}{dt}(m_{b\alpha}\gamma_\alpha v_\alpha(t)) = e_\alpha\Big(E_\varphi(q_\alpha(t),t) + c^{-1}v_\alpha(t)\wedge B_\varphi(q_\alpha(t),t)\Big),\ \ 1\le\alpha\le N,$$
$$c^{-1}\partial_t B = -\nabla\wedge E,\quad c^{-1}\partial_t E = \nabla\wedge B - c^{-1}j,\quad \nabla\cdot E = \rho,\quad \nabla\cdot B = 0,$$
$$\rho(x,t) = \sum_{\alpha=1}^N e_\alpha\varphi(x-q_\alpha(t)),\quad j(x,t) = \sum_{\alpha=1}^N e_\alpha\varphi(x-q_\alpha(t))v_\alpha(t),$$

where $v_\alpha(t) = \dot q_\alpha(t)$, $\gamma_\alpha = (1-(v_\alpha/c)^2)^{-1/2}$, and $m_{b\alpha}$ and e_α denote the bare mass and charge of the α'th particle, respectively. In order to avoid infinities of the self-energy of the particles, they are smeared out by a smooth form factor $\varphi = \varphi(x)$ of support radius R_φ, so that $E_\varphi(x,t) = \int\varphi(x-x')E(x',t)\,dx'$ and $B_\varphi(x,t) = \int\varphi(x-x')B(x',t)\,dx'$. Due to presence of the rigid charge distribution introduced by φ, the model is no longer covariant. However, as compared to the covariant models proposed in [9, 1], it is much easier accessible analytically. We require that initially the particles are far apart ($\sim\varepsilon^{-1}R_\varphi$) and move slowly ($\sim\varepsilon c$). Our aim is to derive an effective ODE whose solutions, as $\varepsilon\to 0$, very well approximate the full dynamics of the original PDE over long times. In order to do so, the Maxwell equations are rewritten as wave equations, e.g. $\Box E = -(\partial_t j + \nabla\rho)$ taking $c = 1$. If the retarded solution is written out explicitly and put into the Lorentz force, then the terms showing up on the right-hand side are roughly of the form $\sum_{\beta=1}^N\int_0^t ds\,\ldots\rho(\ldots + q_\alpha(t) - q_\beta(s))$. The contribution of $\beta=\alpha$ accounts for the self-force of particle α, whereas the parts stemming from indices $\beta\ne\alpha$ are due to the interaction forces. In the following we consider the self-force only. Passing to the system rescaled by ε as mentioned before, the Taylor expansion of $q_\alpha(t) - q_\alpha(s) = -v_\alpha(t)(t-s) + \frac12\ddot q_\alpha(t)(t-s)^2 - \frac16\dddot q_\alpha(t)(t-s)^3 + \ldots$ can be rigorously justified by estimating the errors [8]. Including only $-v_\alpha(t)(t-s)$ and integrating out the retarded integral $\int_0^t ds\,\ldots$ leads to an order zero effective ODE where the effective motion is governed by a Coulomb potential. To the next order (called the Darwin order), an effective velocity-dependent mass M_α shows up, since now additionally the term $\frac12\ddot q_\alpha(t)(t-s)^2$ has to be taken into account. The radiation order is reached as soon as also $-\frac16\dddot q_\alpha(t)(t-s)^3$ is included into the expansion. After the rescaling, the effective equation can be symbolically represented in the form $M_\alpha\ddot q_\alpha = -\nabla V(q) + \varepsilon\dddot q_\alpha$. It turns out that this equation admits very many undesirable solutions that run off to infinity very fast, the so-called run-away solutions [12]. Moreover, since it is a third-order equation and as it has to be compared to the solutions of the original second-order system, it is not clear how the initial data $\ddot{q}_\alpha(0)$ are to be specified. These problems have been resolved in [8]. It has been shown by using geometric singular perturbation theory that there is a $6N$-dimensional center-like manifold $\mathcal{M}_\varepsilon = \{(q, v, h_\varepsilon(q, v)) : q, v \in \mathbb{R}^{3N}\}$ in the phase space $\mathbb{R}^{9N}$ with the following properties: (i) $\mathcal{M}_\varepsilon$ is invariant under the flow of the effective equation, (ii) solutions not on $\mathcal{M}_\varepsilon$ tend to infinity very fast, (iii) if $\ddot{q}(0) = h_\varepsilon(q(0), \dot{q}(0))$ is chosen as the initial acceleration, then the effective solution approximates the full solution to good accuracy in ε over long times ($\sim \varepsilon^{-1}$), and (iv) there is an energy-like quantity (Schott energy) that decreases along the solutions on $\mathcal{M}_\varepsilon$, making manifest the radiative character of the effective equation. See [11] for much more background on this problem that basically goes back to Dirac and others.

2. Kinetic Models

If the number of charged particles is large, it is appropriate to pass to a kinetic description of the matter. For instance, one can consider the relativistic Vlasov-Maxwell system

$$\begin{aligned}
&\partial_t f^\pm + v \cdot \nabla_x f^\pm \pm (E + c^{-1} v \wedge B) \cdot \nabla_p f^\pm = 0, \\
&c^{-1}\partial_t B = -\nabla \wedge E, \quad c^{-1}\partial_t E = \nabla \wedge B - c^{-1} j, \quad \nabla \cdot E = \rho, \quad \nabla \cdot B = 0, \\
&\rho = \int (f^+ - f^-)\, dp, \quad j = \int v(f^+ - f^-)\, dp, \quad v = (1 + c^{-2}|p|^2)^{-1/2} p,
\end{aligned}$$

for two particle species $\pm$ of opposite unity charge and equal unity mass, the species being described by the phase space densities $f^\pm = f^\pm(t, x, p)$. We intend to derive an effective equation approximating the full system in the limit of slow motion $c \to \infty$. We start with a formal expansion of all quantities in powers of c^{-1}, $f^\pm = f_0^\pm + c^{-1} f_1^\pm + c^{-2} f_2^\pm + c^{-3} f_3^\pm + \ldots$, similarly for E, B, ρ, and j, and also $v = p - (c^{-2}/2)p^2 p + \ldots$. Comparing the powers of c^{-1} yields a hierarchy of equations for the coefficient functions $f_j^\pm$ etc. At the order zero, the Vlasov-Poisson system of plasma physics

$$\partial_t f_0^\pm + p \cdot \nabla_x f_0^\pm \pm E_0 \cdot \nabla_p f_0^\pm = 0, \quad E_0(t, x) = -\int |z|^{-2} \bar{z}\, \rho_0(t, x + z)\, dz,$$

is found, where $\bar{z} = |z|^{-1} z$ and $\rho_0 = \int (f_0^+ - f_0^-)\, dp$. This limit has been made rigorous in [10]. Note that due to the absence of a large data existence result for the Vlasov-Maxwell system, already some effort has to be put into the issue of proving the existence of solutions on some time interval $[0, T]$ that does not shrink as $c \to \infty$. Next it can be observed that one can take $f_{2l+1}^\pm = 0$, $E_{2l+1} = 0$, $B_{2l} = 0$ consistently in the hierarchy of equations for the coefficient functions. Then in [4] it has been shown that by passing to the 'Darwin approximation' $f^{\pm,\mathrm{D}} = f_0^\pm + c^{-2} f_2^\pm$, $E^{\mathrm{D}} = E_0 + c^{-2} E_2$, $B^{\mathrm{D}} = c^{-1} B_1$, the order of accuracy can be improved to $O(c^{-3})$. It comes as a certain surprise that this approach does not work any more at the next order, where radiation effects are known to occur. This is due to the fact that the resulting effective system would still be Hamiltonian, not accounting for the energy loss that is present in the system. To get a clue of how the radiation approximation $f^{\pm,\mathrm{R}}$, E^{R}, B^{R} has to be defined, the energy flux formula $\frac{d}{dt}\,\mathcal{E}_r(t) = \frac{c}{4\pi}\int_{|x|=r} \bar{x}\cdot(B\times E)(t,x)\,d\sigma(x) \sim -\frac{2}{3c^3}\,|\dddot{D}(t)|^2$ over the surface of balls $B_r(0)\subset\mathbb{R}^3$ has rigorously been verified in [6] as $r, c\to\infty$; here $D(t) = \int x\rho_0(t,x)\,dx$ denotes the dipole moment of the Newtonian limit. As suggested in [7], the Vlasov equation for the Newtonian distribution is thus modified by incorporating a small correction into the force term as

$$\partial_t \tilde{f}_0^{\pm} + p\cdot\nabla_x \tilde{f}_0^{\pm} \pm \left(\tilde{E}_0 + \frac{2}{3c^3}\dddot{D}\right)\cdot\nabla_p \tilde{f}_0^{\pm} = 0. \tag{1}$$

The additional term is the generalization of the radiation reaction force used in particle models; see Section 1. We also note that for this system an energy-like quantity $\mathcal{E}_{\mathrm{S}}$ (Schott energy) is decreasing such that $\frac{d}{dt}\,\mathcal{E}_{\mathrm{S}}(t) = -\frac{2}{3c^3}|\dddot{D}(t)|^2$. However, analogously to the case of individual particles, the above Vlasov equation coupled to a Poisson equation for the potential does not yield a well-defined PDE system due to the presence of the third-order derivative in time. Therefore once again one has to pass to some kind of center manifold of the system, with the manifold now being infinite-dimensional. Although so far the existence of this manifold has not been verified in full detail, it is possible to use it formally to guess a certain well-defined approximation $D^{[3]}(t)$ to $\dddot{D}(t)$ such that if this replacement of $\dddot{D}$ by $D^{[3]}$ is made in (1), then a well-defined and globally solvable PDE system is obtained. Moreover, $f^{\pm,\mathrm{R}} = \tilde{f}_0^{\pm} + c^{-2}f_2^{\pm}$, $E^{\mathrm{R}} = \tilde{E}_0 + c^{-2}E_2 + (2/3)c^{-3}D^{[3]}$, $B^{\mathrm{R}} = c^{-1}B_1 + c^{-3}B_3$, can be shown to be an effective approximation of the full solutions to order $\mathcal{O}(c^{-4})$. See [5] for a more detailed review.

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