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Hopf bifurcation and exchange of stability in diffusive media

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Abstract

We consider solutions bifurcating from a spatially homogeneous equilibrium under the assumption that the associated linearization possesses continuous spectrum up to the imaginary axis, for all values of the bifurcation parameter, and that a pair of imaginary eigenvalues crosses the imaginary axis. For a reaction-diffusion-convection system we investigate the nonlinear stability of the trivial solution with respect to spatially localized perturbations, prove the occurrence of a Hopf bifurcation and the nonlinear stability of the bifurcating time-periodic solutions, again with respect to spatially localized perturbations.

1. Introduction and main results

We consider the system

$$\left.\begin{aligned} \partial_t U_1 &= \varDelta U_1 + c\partial_{x_1} U_1 + r_1(x)U_2 + \alpha r_2(x)U_1 + N_1(U_1,U_2), \\ \partial_t U_2 &= \varDelta U_2 + c\partial_{x_1} U_2 - r_1(x)U_1 + \alpha r_2(x)U_2 + N_2(U_1,U_2), \end{aligned}\right\} \qquad (1)$$

where $\varDelta = \sum_{j=1}^d \partial_{x_j}^2$, $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$, $t \in [0,\infty[$, and moreover $U = (U_1,U_2) = (U_1,U_2)(x,t) \in \mathbb{R}^2$. Further, $c > 0$ and $\alpha \in \mathbb{R}$ are parameters, whereas $N_i(U_1,U_2)$ denote the nonlinearities. The functions $r_i = r_i(x)$ are supposed to be spatially localized, i.e., they decay to zero at some exponential rate as $|x| \to \infty$. Our assumptions will be such that for $U \in L^2(\mathbb{R}^d,\mathbb{R}^2)$ the stationary solution $U = 0$ of (1) is only marginally stable, in the sense that the associated linearization $\varLambda_{0,\alpha}$ about $U = 0$ possesses essential spectrum up to the imaginary axis. This operator $\varLambda_{0,\alpha}U = BU + R_\alpha(x)U$ splits into two parts, namely into an x-independent part B plus an x-dependent part $R_\alpha(x)$, respectively given by

$$BU = (\Delta + c\partial_{x_1})U \quad \text{and} \quad R_\alpha(x)U = \begin{pmatrix} \alpha r_2(x) & r_1(x) \\ -r_1(x) & \alpha r_2(x) \end{pmatrix} U. \tag{2}$$

According to a classical result (cf. [10, Theorem A.1]) the essential spectrum of the operator B is $\{-|\xi|^2 + ic\xi_1 : \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d\}$, and it is not affected by adding the relatively compact perturbation $R_\alpha(x)$. However, a number of isolated eigenvalues could be created through the operator $R_\alpha(x)$. In order to formulate our precise assumptions, we introduce a spatial weight which results in a shift of the essential spectrum into the left half plane, whereas the isolated eigenvalues remain unchanged. For $\beta \in \mathbb{R}$ we define the operator $\Lambda_{\beta,\alpha}$ by

$$\Lambda_{\beta,\alpha} W = e^{\beta x_1} \Lambda_{0,\alpha}(W e^{-\beta x_1}),$$

i.e., we have

$$\Lambda_{\beta,\alpha} W = \Delta W + (c - 2\beta)\partial_{x_1} W + ((\beta^2 - c\beta)I + R_\alpha(x))W, \tag{3}$$

with I denoting identity. The spectrum of $\Lambda_{\beta,\alpha}$ consists of essential spectrum to the left of $\{z \in \mathbb{C} : \operatorname{Re} z = \beta^2 - c\beta\}$ and of a number of isolated eigenvalues.

Now we are ready to state our main hypotheses. Here and henceforth we fix $c > 0$ and choose $\beta = c/2$. Since we consider α as the bifurcation parameter we will mostly suppress β in our notation and just write Λ_α instead of $\Lambda_{\beta,\alpha}$.

(H1) *The coefficient functions r_i $(i = 1, 2)$ satisfy $r_i \in C^m(\mathbb{R}^d)$ for some $m \in \mathbb{N}$ with $m > d/2$, and moreover there exists a constant $C > 0$ such that*

$$|\partial^j r_1(x)| + |\partial^j r_2(x)| \le C e^{-c|x|/2}, \quad |j| \le m, \quad x \in \mathbb{R}^d.$$

(H2) *The nonlinearities N_i $(i = 1, 2)$ satisfy $N_i \in C^m(\mathbb{R}^2)$, and moreover there exist $C > 0$ and $p \in \mathbb{N}$ with $p \ge 2$ such that for all $U \in \mathbb{R}^2$ with $|U| \le 1$ we have*

$$|N(U)| \le C|U|^p, \quad \textit{where} \quad N(U) = (N_1(U_1, U_2), N_2(U_1, U_2)).$$

In addition, $N(\overline{U}) = \overline{N(U)}$ for $U \in \mathbb{C}^2$.

(H3) *The functions r_1 and r_2 are chosen in such a way that the following holds. There exist $\alpha_c \in \mathbb{R}$, $\delta_0 > 0$, and $\mu > 0$ such that for $\alpha \in]\alpha_c - \delta_0, \alpha_c + \delta_0[$ all eigenvalues $\lambda \in \mathbb{C}$ of Λ_α, except of two, satisfy $\operatorname{Re} \lambda \le -\mu$. The other two eigenvalues $\lambda_0^\pm(\alpha)$ satisfy*

$$\lambda_0^\pm(\alpha_c) = \pm i\omega_0 \ne 0 \tag{4}$$

with $\omega_0 > 0$*, and*

$$\frac{d}{d\alpha}\operatorname{Re}\lambda_0^{\pm}(\alpha)\Big|_{\alpha=\alpha_c} > 0. \tag{5}$$

Examples of functions r_1 and r_2 such that (H1) and (H3) hold are $r_i(x) = c_i \operatorname{sech}^2(d_i|x|^2)$ for $i = 1, 2$, with constants c_i and d_i properly chosen.

Remark 1. From assumption (H3) we have the following consequences for the spectrum of $\Lambda_{\beta,\alpha}$ for a variable $\beta \in [0, c/2]$. As mentioned before, the spectrum of $\Lambda_{\beta,\alpha}$ consists of essential spectrum to the left of $\{z \in \mathbb{C} : \operatorname{Re} z = \beta^2 - c\beta\}$ and of a number of isolated eigenvalues. If W_0 is an eigenfunction of $\Lambda_\alpha = \Lambda_{c/2,\alpha}$ with eigenvalue λ_0, then a straightforward calculation using (3) shows that $W_1(x) = e^{(\beta - c/2)x_1} W_0(x)$ is an eigenfunction of $\Lambda_{\beta,\alpha}$ corresponding to the same eigenvalue λ_0. Therefore the isolated eigenvalues of $\Lambda_{\beta,\alpha}$ are independent of β until they vanish in the essential spectrum. We note that this reasoning should be considered as being only formal, since no spaces are specified to which the notion of an eigenvalue is linked. However, no explicit use of this remark will be made in the sequel.

For future applications we also consider another class of amplifications and nonlinear terms which are of Navier-Stokes type.

(H1)' *The coefficient functions* r_i *have the form* $r_i(x) = \partial_{x_l}\tilde{r}_i(x)$ *for* $i = 1, 2$*, with some* $l \in \{1, \ldots, d\}$ *and functions* $\tilde{r}_i \in C^m(\mathbb{R}^n)$*. Moreover, there exists a constant* $C > 0$ *such that*

$$|\partial_x^j \tilde{r}_1(x)| + |\partial_x^j \tilde{r}_2(x)| \le C e^{-c|x|/2}, \quad |j| \le m, \quad x \in \mathbb{R}^d.$$

(H2)' *The nonlinearity* $N = (N_1, N_2)$ *has the form*

$$N(U) = \sum_{j=1}^{d} \partial_{x_j} N^j(U),$$

with functions $N^j \in C^m(\mathbb{R}^2, \mathbb{R}^2)$ *for* $j = 1, \ldots, d$*. Moreover, there exist* $C > 0$ *and* $p \in \mathbb{N}$ *with* $p \ge 2$ *such that for all* $U \in \mathbb{R}^2$ *with* $|U| \le 1$ *we have*

$$|N^j(U)| \le C|U|^p, \qquad \text{where} \quad N^j(U) = (N_1^j(U_1, U_2), N_2^j(U_1, U_2)).$$

In addition, $N^j(\overline{U}) = \overline{N^j(U)}$ *for* $U \in \mathbb{C}^2$*.*

Using the incompressibility condition, the nonlinear terms of the Navier-Stokes equations can be cast into the form considered in (H2)'. In contrast to (H2)' it is not obvious whether the technical assumption (H1)', which is needed in Section 3 and which implies (H1), can be satisfied in applications. See also Section 4.

Assumption (H3) is reminiscent of the assumption for the classical Hopf bifurcation, and so the purpose of this paper is to investigate the bifurcation

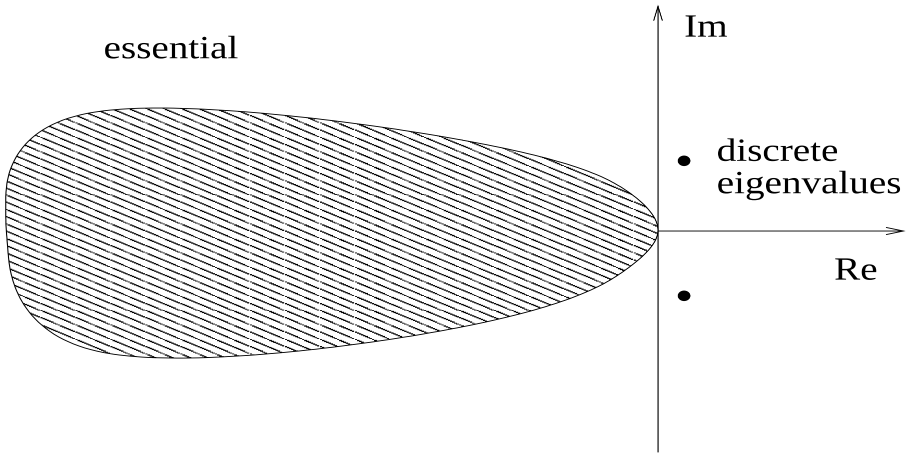

Fig. 1. Spectrum of the linearization $\Lambda_{0,\alpha}$ about the trivial solution $U = 0$

scenario of (1) in a neighborhood of $U = 0$ for α close to α_c. The new difficulty which occurs here is due to the fact that the linearization $\Lambda_{0,\alpha}$ about $U = 0$ possesses continuous spectrum up to the imaginary axis, without any spectral gap, as is indicated in Figure 1.

Therefore the nonlinear stability of $U = 0$ for $\alpha < \alpha_c$, the occurrence of a Hopf bifurcation at $\alpha = \alpha_c$, and the exchange of stability from $U = 0$ to the bifurcating time-periodic solutions are not clear at all.

Our first theorem concerns the stability of the trivial solution $U = 0$ for $\alpha < \alpha_c$. We prove the nonlinear stability with respect to spatially localized perturbations.

Theorem 1. *Assume $c > 0$ is fixed, and (H1), (H2) or (H2)', and (H3) are satisfied. If $p > 1 + 2/d$ for (H2) or if $p > 2/d$ for (H2)' and $\alpha < \alpha_c$, then there exists $\gamma > 0$ such that for every $\delta_1 > 0$ one may choose $\delta_2 > 0$ with the property that*

$$\|U(\cdot, 0)\|_{L^\infty} + \|U(\cdot, 0)\|_{L^1} + \|W(\cdot, 0)\|_{L^1 \cap L^\infty} \leq \delta_2$$

for the initial data implies

$$\sup_{t \in [0,\infty[} \Big((1 + t)^{d/2} \|U(\cdot, t)\|_{L^\infty} + \|U(\cdot, t)\|_{L^1} + e^{\gamma t} \|W(\cdot, t)\|_{L^1 \cap L^\infty} \Big) \leq \delta_1$$

for the solution U of (1), where $W(x, t) = U(x, t)e^{\beta x_1}$ with $\beta = c/2$.

The proof, which is elaborated in Section 2, is based upon the following facts. The term $R_\alpha(x)U$ in (1) leads to an amplification of perturbations of $U = 0$ near $x = 0$. This is counter-acted by the effect of the drift term $c\partial_{x_1} U$ which tends to transport these perturbations away from $x = 0$ to infinity. Thus after a sufficiently long time the diffusion term will dominate and force the perturbations to decay with rate $t^{-d/2}$. Generally speaking, all nonlinear terms are asymptotically irrelevant with respect to diffusion, i.e., the nonlinear system

$$\partial_t U = \Delta U + N(U), \quad U|_{t=0} = U(\cdot, 0),$$

with $N = (N_1, N_2)$ shows the same asymptotic behavior as the linear diffusion equation, for sufficiently small and sufficiently spatially localized initial data $U(\cdot, 0)$; see e.g. [3]. From a technical point of view, to recover the diffusive behavior behind the amplification we introduce suitable norms with exponential weights in space. For perturbations which are small in these norms the influence of the spatially localized amplification vanishes exponentially in time, since the distance between the main perturbation and the spatial amplification at $x = 0$ grows linearly; see [6] and the references therein.

At $\alpha = \alpha_c$ two complex conjugate eigenvalues with nonvanishing imaginary part cross the imaginary axis. Hence the trivial solution becomes unstable. Although we have continuous spectrum up to the imaginary axis a finite dimensional reduction is possible, and like in the case of a classical generic Hopf bifurcation the problem of bifurcating time-periodic solutions is reduced to the solution of an equation of the form

$$G(\alpha - \alpha_c, r) = \kappa_1(\alpha - \alpha_c)r + \kappa_3 r^3 + \text{h.o.t.} = 0,$$

with a smooth function $G : \mathbb{R}^2 \to \mathbb{R}$ and coefficients $\kappa_1, \kappa_3 \in \mathbb{R}$ related to the nonlinearity; cf. Remark 3 below for more precise information, in particular for the definition of the coefficient κ_3. Depending on the sign of κ_3 the next theorem guarantees the occurrence of a supercritical or a subcritical Hopf bifurcation. It is formulated only for the non-degenerated case, i.e., $\kappa_3 \neq 0$.

Theorem 2. *Assume $c > 0$ is fixed, $\kappa_3 \neq 0$, and (H1), (H2) or (H1)', (H2)', and (H3) are satisfied. If $d \geq 4$ for (H1), (H2) or if $d \geq 1$ for (H1)', (H2)', then (1) has a one-dimensional family of small time-periodic solutions, i.e., there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$ we have*

$$U_{\text{per}}(x, t) = U_{\text{per}}(x, t + 2\pi/\omega)$$

solving (1) for $\alpha = \alpha_c - \text{sgn}(\kappa_3)\varepsilon$. Moreover,

$$\omega = \omega_0 + \mathcal{O}(\varepsilon) \quad \textit{and} \quad \|U_{\text{per}}\|_{C_b^0(\mathbb{R}^d \times [0, 2\pi/\omega])} = \mathcal{O}(\sqrt{\varepsilon}).$$

If in addition the hypothesis

(H4) *System (1) is equivariant (cf. [8]) under $U \mapsto -U$*

holds, then the assertion of the theorem is true regardless of the space dimension $d \geq 1$.

The proof is given in Section 3. The periodic solutions can be written as convergent series

$$U_{\mathrm{per}}(x,t) = \sum_{n\in\mathbb{Z}} U_n(x) e^{in\omega t},$$

where for $n \neq 0$ the functions U_n vanish with some uniform (in n) exponential rate as $|x| \to \infty$, whereas in general $U_0 \in H^m$ is not decaying exponentially. However, in the important special case that the equivariance condition (H4) is satisfied, we have $U_0 = 0$ for symmetry reasons; see Lemma 1 below. An example of a system (1) which is equivariant in the sense of (H4) is provided by the nonlinearity

$$N_1(U_1,U_2) = -U_1(|U_1|^2 + |U_2|^2) \quad \text{and} \quad N_2(U_1,U_2) = -U_2(|U_1|^2 + |U_2|^2), \tag{6}$$

which also satisfies (H2) with $p = 3$, whence Theorem 2 applies.

The proof of Theorem 2 is again based on an interplay of the (spatial) uniform norm and an exponentially weighted norm. This interplay allows us to use the classical Lyapunov-Schmidt method. The difficulties with the continuous spectrum touching the imaginary axis then become evident in an equation for U_0 (the mean value in time) to be solved, which has the form

$$\varDelta U_0 + c\partial_{x_1} U_0 = f$$

in the case of (H2), and which reads as

$$\varDelta U_0 + c\partial_{x_1} U_0 = \partial_{x_j} f$$

in the case of (H2)'. Although the spectrum of the operator on the left-hand side touches the imaginary axis, these equations admit a unique solution, if (H2) holds in at least four space dimensions, respectively if (H2)' holds in any space dimension. In the Lyapunov-Schmidt procedure the function f will contain the nonlinear terms, and it will turn out that additionally $f \in L^1$ is satisfied, a fact that will be very useful. According to the above remarks the difficulties in solving these equations do not occur if (H4) is assumed, since then $U_0 = 0$.

The presence of a "background field" U_0 of U_{per} makes the proof of the exchange of stability from the trivial solution $U = 0$ to the time-periodic solution U_{per} more delicate. As mentioned before, the time-dependent parts of U_{per} decay with some exponential rate in space. This allows us to handle these terms once more by introducing a suitable exponential weight in space, and hence in the case that $U_0 = 0$ the exchange of stability works as for the classical Hopf bifurcation. However, the time-independent part U_0 in general only decays at a polynomial rate and cannot be dealt with in the same manner. But nevertheless the part associated to U_0 is still a relatively compact perturbation which can be handled by well known estimates for linear Schrödinger operators. These estimates are very similar to the estimates which we use in the proof of Theorem 1.

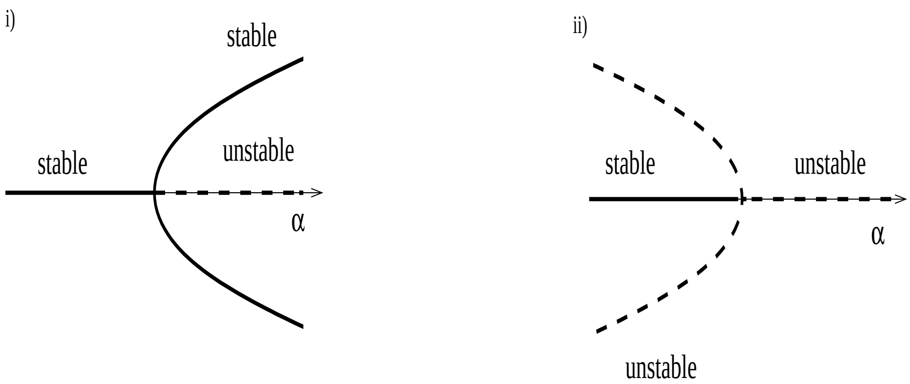

Fig. 2. Exchange of stability in the classical and diffusive Hopf bifurcation, i) if $\kappa_3 < 0$ and ii) if $\kappa_3 > 0$.

Hence the classical exchange of stability can be established if $\kappa_3 < 0$.

Theorem 3. *Assume $c > 0$ is fixed, $\kappa_3 < 0$, and (H1), (H2) for $d \geq 4$ or (H1)', (H2)' for $d \geq 2$, and (H3) are satisfied. Then there exists $\varepsilon_0 > 0$ such that for all $\alpha \in]\alpha_c, \alpha_c + \varepsilon_0[$ the time-periodic solution $U_{\mathrm{per}}(x,t)$ is stable with respect to spatially localized perturbations. More precisely, for $\alpha \in]\alpha_c, \alpha_c + \varepsilon_0[$ there exists $\beta_0 > 0$ such that for all $\beta \in]0, \beta_0[$ there is $\sigma > 0$ such that the following is true. For all $\delta_1 > 0$ there exists $\delta_2 > 0$ such that for the solutions $U = U_{\mathrm{per}} + V$ of (1) we have*

$$\sup_{t \in [0,\infty[} \left((1+t)^{d/2} \|V(\cdot,t)\|_{L^\infty} + \|V(\cdot,t)\|_{L^1} + e^{\sigma t} \|W(\cdot,t)\|_{L^1 \cap L^\infty} \right) \leq \delta_1,$$

provided the initial data satisfy

$$\|V(\cdot,0)\|_{L^\infty} + \|V(\cdot,0)\|_{L^1} + \|W(\cdot,0)\|_{L^1 \cap L^\infty} \leq \delta_2,$$

where $W(x,t) = V(x,t) e^{\beta x_1}$.

The proof is carried out in Section 4.

There are a number of physical problems where the situation occurs which is considered here. Examples are the flow around a body in the Navier-Stokes equation [1,2] or the bifurcations of a tip of a spiral wave [17]. It is the purpose of this paper to come closer to a mathematical understanding of the bifurcation scenario in such systems. The bifurcation scenario is delicate due to the fact that for all values of the bifurcation parameter we have essential spectrum up to the imaginary axis. Therefore classical reduction methods (which are available if the spectrum of the linearization separates into a finite-dimensional part close to the imaginary axis and an infinite-dimensional part which lies strictly in the left half-plane of the complex plane) as the Lyapunov-Schmidt method and the center manifold theorem are no longer available, at least in the classical set-up.

As a first step towards an understanding of this question we considered in [14] a nonlinear diffusion equation in one space dimension, in the case of a real eigenvalue crossing the imaginary axis. There we proved the nonlinear stability of the trivial solution with respect to spatially localized perturbations if there is no eigenvalue in the right half plane, the occurrence of a pitchfork bifurcation in case of the real eigenvalue crossing the imaginary axis, and finally an exchange of stability from the trivial solution to the bifurcating equilibrium.

Investigations of bifurcation scenarios including continuous spectrum can also be found in a number of other papers, see for instance [4,9,11–13,15,18] and the references therein.

Notation. Throughout the paper different irrelevant constants are denoted by the same symbol C. For simplicity we often write U^k, although $U \in \mathbb{R}^2$. Here U^k stands for any of the terms $U_1^{k_1} U_2^{k_2}$ with $k_1 + k_2 = k$. Mostly we will be not too precise about the particular form of the nonlinearity N, since this would lead to much additional notation. However, for definiteness always the nonlinearity from (6) should be kept in mind.

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2. Nonlinear stability of the trivial solution for $\alpha < \alpha_c$

It is the purpose of this section to verify for $\alpha < \alpha_c$ the stability of the trivial solution $U = 0$ with respect to spatially localized perturbations. Although only a suitable modification of the proof given in [14, Thm. 2.1], the details are included since the method will be used again in Section 4. However, for simplicity we will restrict ourselves to nonlinearities satisfying (H2).

The result will be established by simultaneously controlling the solution U of (1) in three different norms. In order to estimate the influence of the amplification, we rely on a weighted norm in space and introduce $W(x,t) = U(x,t)e^{\beta x_1}$, with $\beta = c/2$. Then (1) and (3) yield

$$\partial_t W = \Lambda_\alpha W + F(U, W),$$

with $F(U,W) = e^{\beta x_1} N(U)$ being a smooth nonlinear mapping, where we understand that in $N(U)$ each factor U could be replaced by $e^{-\beta x_1} W$ if desired. For instance, if N is a polynomial, then we replace every U^k in $N(U)$ by $U^{k-1} e^{-\beta x_1} W$, which results in the contribution $U^{k-1} W$ to $F(U,W)$. In particular, (H2) yields

$$|F(U(x), W(x))| \leq C|U(x)|^{p-1}|W(x)|. \tag{7}$$

The choice of F is not unique, but for the rest of the paper just one such F is fixed.

Example 1. For the nonlinearity from (6) we let

$$F(U,W) = \begin{pmatrix} -W_1(|U_1|^2 + |U_2|^2) \\ -W_2(|U_1|^2 + |U_2|^2) \end{pmatrix}$$

for $U = (U_1, U_2)$ and $W = (W_1, W_2)$.

Conversely, the variable W will be used to replace U in (1) where it appears with an exponentially vanishing factor; at this point we use the fact that R_α decays to zero at an exponential rate as $|x| \to \infty$. Hence we are led to consider the augmented system

$$\left.\begin{aligned} \partial_t U &= BU + e^{-\beta x_1} R_\alpha(x) W + N(U), \\ \partial_t W &= \Lambda_\alpha W + F(U,W). \end{aligned}\right\} \tag{8}$$

For this system we need to derive bounds both in the $\|\cdot\|_{L^\infty}$-norm and in the $\|\cdot\|_{L^1}$-norm; we make this evident in our notation by denoting the same solution U by $u = U$, if $u \in C_b^0 = C_b^0(\mathbb{R}^d, \mathbb{R}^2)$, and by $v = U$, if $v \in L^1 = L^1(\mathbb{R}^d, \mathbb{R}^2)$. Then u, v, and $w = W$ are a solution of the system

$$\left.\begin{aligned} \partial_t u &= Bu + e^{-\beta x_1} R_\alpha(x) w + N(u), \\ \partial_t v &= Bv + e^{-\beta x_1} R_\alpha(x) w + N_1(u,v), \\ \partial_t w &= \Lambda_\alpha w + F(u,w), \end{aligned}\right\} \tag{9}$$

where $N_1(u,v)$ is a smooth nonlinear mapping which is obtained from $N(u)$ in the same way as $F(u,w)$ is obtained from $N(u)$, e.g., u^k in $N(u)$ will be replaced by $u^{k-1}v$. Hence we also have

$$|N_1(u(x), v(x))| \leq C|u(x)|^{p-1}|v(x)|. \tag{10}$$

Although not explicitly stated we will also use the v-variable in the first equation of (9). To get a clue on what kind of behavior may be expected for the diffusive variables u and v, we note that $\tilde{u}(\xi,t) = u(\xi - cte_1, t)$ and $\tilde{v}(\xi,t) = v(\xi - cte_1, t)$ satisfy (with $e_1 = (1,0,\ldots,0)$) the system

$$\left.\begin{aligned} \partial_t \tilde{u} &= \Delta \tilde{u} + N(\tilde{u}) + \ldots, \\ \partial_t \tilde{v} &= \Delta \tilde{v} + N_1(\tilde{u}, \tilde{v}) + \ldots, \end{aligned}\right\} \tag{11}$$

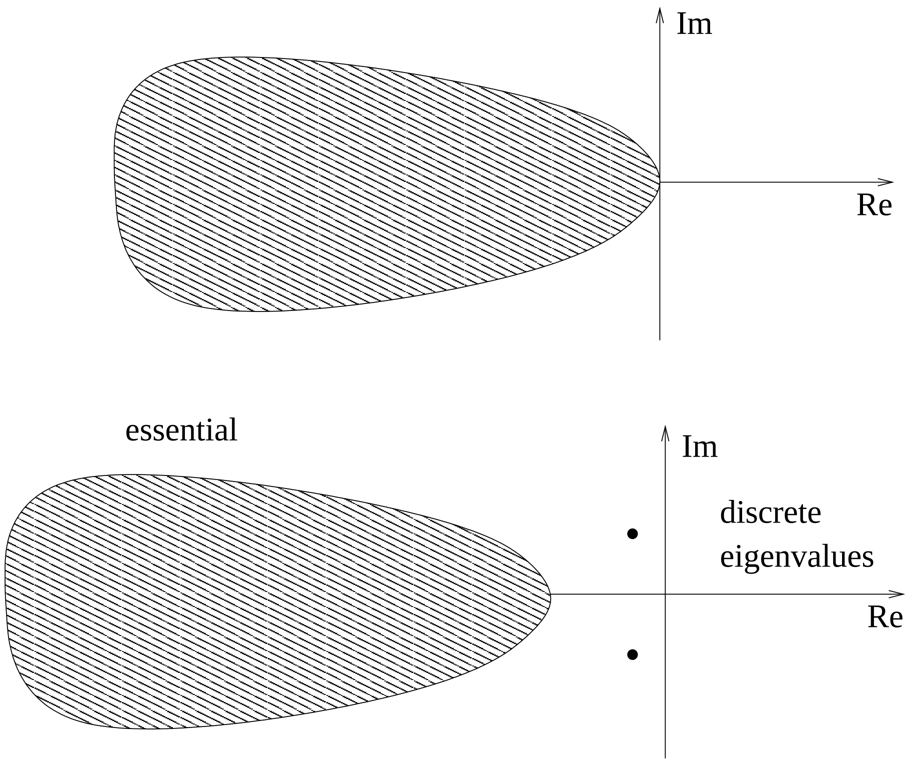

Fig. 3. Spectra of the linearization $B + e^{-\beta x_1} R_\alpha(x) I$ and $\Lambda_{\beta,\alpha}$ in (8) about the trivial solution $U = 0$.

the dots indicating terms that vanish at an exponential rate. For (11) it is known that the diffusive behavior of the linearized system can be used to control the nonlinear terms and to prove diffusive behavior also for the nonlinear system. Hence we expect, for sufficiently small spatially localized perturbations, the u-variable in (9) to decay as $\mathcal{O}(t^{-d/2})$, the v-variable to remain bounded, and accordingly the w-variable to decay at an exponential rate. These arguments are made rigorous in the following

Proof of Theorem 1 : We introduce

$$
\begin{aligned}
a(t) &= \sup_{s\in[0,t]} \Big((1+s)^{d/2} \|u(s)\|_{L^\infty} \Big), \\
b(t) &= \sup_{s\in[0,t]} \|v(s)\|_{L^1}, \\
c(t) &= \sup_{s\in[0,t]} \Big(e^{\gamma s} \|w(s)\|_{L^1\cap L^\infty} \Big),
\end{aligned}
$$

where $u(s)$ denotes $u(\cdot, s)$, etc., and $\gamma > 0$ is specified below in (13). From the variation of constants formula and (9) we obtain

$$u(t) = e^{Bt}u_0 + \int_0^t e^{B(t-s)}\left[e^{-\beta x_1}R_\alpha(x)w(s) + N(u(s))\right] ds,$$
$$v(t) = e^{Bt}u_0 + \int_0^t e^{B(t-s)}\left[e^{-\beta x_1}R_\alpha(x)w(s) + N_1(u(s), v(s))\right] ds,$$
$$w(t) = e^{\Lambda_\alpha t}w_0 + \int_0^t e^{\Lambda_\alpha(t-s)}[F(u(s), w(s))]\, ds,$$

with $w_0(x) = e^{\beta x_1}u_0(x)$. Next we remark that

$$\|e^{Bt}\|_{L^1\to L^1} \le C, \quad \|e^{Bt}\|_{L^1\to L^\infty} \le Ct^{-d/2}, \quad \|e^{Bt}\|_{L^\infty\to L^\infty} \le C, \quad (12)$$
$$\|e^{\Lambda_\alpha t}\|_{L^1\cap L^\infty\to L^1\cap L^\infty} \le Ce^{-2\theta t},$$

for suitable constants $C > 0$ and $\theta > 0$. The first three estimates are due to the fact that the solution of $\partial_t u = Bu = \Delta u + c\partial_{x_1}u$ with initial data $u|_{t=0} = u_0$ is given by $u(x,t) = u_{\mathrm{diffus}}(x + cte_1, t)$, where $e_1 = (1, 0, \dots, 0)$ and

$$u_{\mathrm{diffus}}(x,t) = \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} u_0(y)dy.$$

Further, the operator Λ_α is sectorial. Hence the bound on $e^{\Lambda_\alpha t}$ is obtained with

$$\theta = \frac{1}{3}\min\left\{c^2/4,\ \mu,\ |\mathrm{Re}\,\lambda_0^+(\alpha)|,\ |\mathrm{Re}\,\lambda_0^-(\alpha)|\right\} > 0,$$

since the essential spectrum of Λ_α lies to the left of $\{z \in \mathbb{C} : \mathrm{Re}\, z = \beta^2 - c\beta = -c^2/4\}$, and by (H3) the eigenvalues except of two are located to the left of $\{z \in \mathbb{C} : \mathrm{Re}\, z = -\mu\}$, and for the remaining two eigenvalues we moreover have $\mathrm{Re}\,\lambda_0^\pm(\alpha) < 0$ by (4) and (5). With this choice of θ we moreover set

$$\gamma = \theta. \quad (13)$$

First we estimate

$$\begin{aligned} &e^{\gamma t}\|w(t)\|_{L^1\cap L^\infty} \\ &\le e^{\gamma t}\|e^{\Lambda_\alpha t}w_0\|_{L^1\cap L^\infty} \\ &\quad + e^{\gamma t}\int_0^t \|e^{\Lambda_\alpha(t-s)}\|_{L^1\cap L^\infty\to L^1\cap L^\infty}\|F(u(s), w(s))\|_{L^1\cap L^\infty}\, ds. \end{aligned}$$

As a consequence of (7) we see that

$$|F(u(x,s), w(x,s))| \le C|u(x,s)|^{p-1}|w(x,s)|,$$

and this yields $\|F(u(s),w(s))\|_{L^1\cap L^\infty} \leq C\|u(s)\|_{L^\infty}^{p-1}\|w(s)\|_{L^1\cap L^\infty}$. Hence we obtain

$$\begin{aligned} e^{\gamma t}\|w(t)\|_{L^1\cap L^\infty} &\leq Ce^{(\gamma-2\theta)t}\|w_0\|_{L^1\cap L^\infty} \\ &\quad + Ce^{\gamma t}\int_0^t e^{-2\theta(t-s)}\|u(s)\|_{L^\infty}^{p-1}\|w(s)\|_{L^1\cap L^\infty}\,ds \\ &\leq Cc(0) \\ &\quad + Ca(t)^{p-1}c(t)\int_0^t e^{(\gamma-2\theta)(t-s)}(1+s)^{-d(p-1)/2}\,ds \\ &\leq C\Big(c(0)+a(t)^{p-1}c(t)\Big). \end{aligned}$$

Taking the supremum over time, it follows that

$$c(t) \leq C\Big(c(0)+a(t)^{p-1}c(t)\Big), \tag{14}$$

and here no assumption on $p \geq 1$ is needed, since all nonlinear terms can be controlled by the exponential decay of the linear semigroup $e^{\Lambda_\alpha t}$.

Next we invoke (H1) and $\beta = c/2$ to get

$$|e^{-\beta x_1}R_\alpha(x)| \leq Ce^{c|x|/2}e^{-c|x|/2} \leq C,$$

and thus (10) yields

$$\begin{aligned} \|v(t)\|_{L^1} &\leq \|e^{Bt}\|_{L^1\to L^1}\|u_0\|_{L^1} \\ &\quad + \int_0^t \|e^{B(t-s)}\|_{L^1\to L^1}\|e^{-\beta x_1}R_\alpha(x)w(s)\|_{L^1}\,ds \\ &\quad + \int_0^t \|e^{B(t-s)}\|_{L^1\to L^1}\|N_1(u(s),v(s))\|_{L^1}\,ds \\ &\leq Cb(0) + C\int_0^t \|w(s)\|_{L^1\cap L^\infty}\,ds \\ &\quad + C\int_0^t \|u(s)\|_{L^\infty}^{p-1}\|v(s)\|_{L^1}\,ds \\ &\leq Cb(0) + Cc(t)\int_0^t e^{-\gamma s}\,ds \\ &\quad + Ca(t)^{p-1}b(t)\int_0^t (1+s)^{-d(p-1)/2}\,ds. \end{aligned}$$

Therefore

$$b(t) \leq C\Big(b(0)+a(t)^{p-1}b(t)+c(t)\Big), \tag{15}$$

since $-d(p-1)/2 < -1$ by assumption. In addition, using (H2) and $u(s) = v(s)$ we deduce

$$
\begin{aligned}
&(1+t)^{d/2}\|u(t)\|_{L^\infty} \\
&\leq (1+t)^{d/2}\min\Big\{\|e^{Bt}\|_{L^1\to L^\infty}\|u_0\|_{L^1}, \|e^{Bt}\|_{L^\infty\to L^\infty}\|u_0\|_{L^\infty}\Big\} \\
&\quad +(1+t)^{d/2}\int_0^t \min\Big\{\|e^{B(t-s)}\|_{L^1\to L^\infty}\|N(u(s))\|_{L^1}, \\
&\qquad\qquad\qquad\qquad \|e^{B(t-s)}\|_{L^\infty\to L^\infty}\|N(u(s))\|_{L^\infty}\Big\}\, ds \\
&\quad +(1+t)^{d/2}\int_0^t \min\Big\{\|e^{B(t-s)}\|_{L^1\to L^\infty}\|e^{-\beta x_1}R_\alpha(x)w(s)\|_{L^1}, \\
&\qquad\qquad\qquad\qquad \|e^{B(t-s)}\|_{L^\infty\to L^\infty}\|e^{-\beta x_1}R_\alpha(x)w(s)\|_{L^\infty}\Big\}\, ds \\
&\leq C\Big(a(0)+b(0)\Big) \\
&\quad + C(1+t)^{d/2}\int_0^t \min\Big\{(t-s)^{-d/2}\|v(s)\|_{L^1}, \|u(s)\|_{L^\infty}\Big\}\, \|u(s)\|_{L^\infty}^{p-1}\, ds \\
&\quad + C(1+t)^{d/2}\int_0^t \min\Big\{(t-s)^{-d/2}, 1\Big\}\, \|w(s)\|_{L^1\cap L^\infty}\, ds \\
&\leq C\Big(a(0)+b(0)\Big) \\
&\quad + C\Big(a(t)+b(t)\Big)a(t)^{p-1} \\
&\qquad \times (1+t)^{d/2}\int_0^t \min\Big\{(t-s)^{-d/2}, (1+s)^{-d/2}\Big\}\,(1+s)^{-d(p-1)/2}\, ds \\
&\quad + Cc(t)\,(1+t)^{d/2}\int_0^t \min\Big\{(t-s)^{-d/2}, 1\Big\}\, e^{-\gamma s}\, ds \\
&\leq C\Big(a(0)+b(0)\Big) + C\Big(a(t)+b(t)\Big)a(t)^{p-1} + Cc(t),
\end{aligned}
$$

the latter since due to $d(p-1)/2 > 1$ the factors are bounded functions of t. Thus we obtain

$$
a(t) \leq C\Big(a(0)+b(0)+[a(t)+b(t)]a(t)^{p-1}+c(t)\Big). \tag{16}
$$

Altogether, we have shown in (16), (15), and (14) that for $t \in [0,\infty[$ the estimates

$$
\left.
\begin{aligned}
a(t) &\leq C_1\Big(a(0)+b(0)+a(t)^p+a(t)^{p-1}b(t)+c(t)\Big), \\
b(t) &\leq C_1\Big(b(0)+a(t)^{p-1}b(t)+c(t)\Big), \\
c(t) &\leq C_1\Big(c(0)+a(t)^{p-1}c(t)\Big),
\end{aligned}
\right\} \tag{17}
$$

are satisfied, for a constant $C_1 > 0$. Analogously to the proof of [14, Thm. 2.1] this implies that given $\delta_1 > 0$ we may choose $\delta_2 > 0$ such that $a(0)+b(0)+c(0) \leq \delta_2$ leads to $a(t)+b(t)+c(t) \leq \delta_1$ for all $t \in [0,\infty[$. This finishes the proof of Theorem 1. □

3. Hopf bifurcation at $\alpha = \alpha_c$

We consider $\omega > 0$ close to ω_0 from (4) in (H3), and we look for $2\pi/\omega$-time-periodic solutions of

$$\partial_t U = BU + R_\alpha(x)U + N(U), \tag{18}$$

using the notation introduced in Section 1. First we assume that (H2) is satisfied for the nonlinearity. By considering $V(x,t) = U(x,t/\omega)$, and writing again U for V, we see that we need to find 2π-time-periodic solutions of

$$\omega\partial_t U = BU + R_\alpha(x)U + N(U).$$

We proceed as in Section 2, cf. (8), and deal with the augmented system

$$\left.\begin{aligned} \omega\partial_t U &= BU + e^{-\beta x_1} R_\alpha(x)W + N(U), \\ \omega\partial_t W &= \Lambda_\alpha W + F(U,W). \end{aligned}\right\} \tag{19}$$

Due to the periodicity in time we make the ansatz

$$U(x,t) = \sum_{n\in\mathbb{Z}} U_n(x)e^{int} \quad \text{and} \quad W(x,t) = \sum_{n\in\mathbb{Z}} W_n(x)e^{int}.$$

In view of the spaces which will be chosen below for the coefficient functions U_n and W_n, these series will in particular be uniformly convergent on $\mathbb{R}^d \times [0,2\pi]$; see Lemma 2. From (19) we then obtain the system

$$\left.\begin{aligned} (in\omega I - B)U_n &= e^{-\beta x_1} R_\alpha(x)W_n + N_n(U), \\ (in\omega I - \Lambda_\alpha)W_n &= F_n(U,W), \end{aligned}\right\} \tag{20}$$

for $n \in \mathbb{Z}$, where we set

$$N(U)(x,t) = \sum_{n\in\mathbb{Z}} N_n(U)(x)\, e^{int}, \quad F(U,W)(x,t) = \sum_{n\in\mathbb{Z}} F_n(U,W)(x)\, e^{int}. \tag{21}$$

Since we are interested in real-valued solutions only, we will always suppose that $\overline{U_n} = U_{-n}$ and $\overline{W_n} = W_{-n}$ for $n \in \mathbb{Z}$. An obvious remark is

Lemma 1. *If system (1) satisfies the equivariance condition (H4), then the subspace of time-periodic solutions which possess an expansion*

$$U(x,t) = \sum_{n\in 2\,\mathbb{Z}+1} U_n(x)e^{int} \quad \text{and} \quad W(x,t) = \sum_{n\in 2\,\mathbb{Z}+1} W_n(x)e^{int}$$

is invariant. Therefore for such systems no even indices occur, in particular $U_0 = 0$.

We would like to apply the Lyapunov-Schmidt method to (19), and hence we consider the linearized system first. Consequently we have to investigate the existence of $(B - in\omega I)^{-1}$ and $(\Lambda_\alpha - in\omega I)^{-1}$. By the assumptions on the spectrum, the second inverse exists for $n \in \mathbb{Z} \setminus \{-1, 1\}$, and the first inverse exists for $n \in \mathbb{Z} \setminus \{0\}$. In the case that $n \in \{-1, 1\}$ and $\omega = \omega_0$, each of the operators $\Lambda_{\alpha_c} \pm i\omega_0 I$ possesses a one-dimensional kernel, i.e., in sum there is a two-dimensional kernel. As already explained in the introduction our analysis is based on the fact that the operator B can be inverted from $L^2 \cap L^1$ to H^2 if the x-variable lives in $\mathbb{R}^d$ with $d \geq 4$ and if (H2) is valid, although B has continuous spectrum up to the imaginary axis touching zero. Thus, as for the classical Hopf bifurcation, the Lyapunov-Schmidt method will lead to a two-dimensional reduced system.

In order to make these arguments rigorous, we need some function spaces. Due to their different decay properties, we distinguish between the mode $n = 0$ with coefficient $U_0(x)$ serving as a "background field", and the other modes $n \neq 0$ with exponentially fast decaying coefficients $U_n(x)$. Contrary to that, the weighted variables $W_n(x)$ decay exponentially for all $n \in \mathbb{Z}$. As a consequence, U_0 will decay at some exponential rate before the potential R_α, whereas behind the potential we can only expect polynomial decay rates.

We denote by $H^m = H^m(\mathbb{R}^d, \mathbb{C}^2)$ the usual Sobolev space, and by H^m_γ the spatially weighted Sobolev space equipped with the norm

$$\|u\|_{H^m_\gamma} = \|u\rho_\gamma\|_{H^m},$$

where $\rho_\gamma(x) = e^{\gamma\sqrt{1+|x|^2}}$; this weight forces the functions in H^m_γ to decay to zero like $e^{-\gamma|x|}$ for $|x| \to \infty$. The Fourier transform is an isomorphism between the spaces H^m and L^2_m, where the latter is defined via the norm $\|u\|_{L^2_m} = \|u\rho^m\|_{L^2}$, with $\rho(x) = \sqrt{1 + |x|^2}$. We also recall that $\beta = c/2$ has been fixed, and we will take $0 < \gamma \ll \beta$, cf. Lemma 7 below.

Before going on to the proof of Theorem 2, we need to collect some preliminary results. For a fixed $m \in \mathbb{N}$ such that $m > d/2$ we will solve (20) for $W = (W_n)_{n\in\mathbb{Z}}$ in the space

$$\mathcal{Y} = \{\tilde{u} = (\tilde{u}_n)_{n\in\mathbb{Z}} : \|\tilde{u}\|_{\mathcal{Y}} < \infty\}$$

with the norm defined by

$$\|\tilde{u}\|_{\mathcal{Y}} = \sum_{n\in\mathbb{Z}} \|\tilde{u}_n\|_{H^m_\gamma},$$

whereas $U = (U_n)_{n\in\mathbb{Z}}$ will lie in the space

$$\mathcal{Z} = \{\tilde{u} = (\tilde{u}_n)_{n\in\mathbb{Z}} : \|\tilde{u}\|_{\mathcal{Z}} < \infty\},$$

equipped with the norm

$$\|\tilde{u}\|_{\mathcal{Z}} = \|\tilde{u}_0\|_{H^m} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \|\tilde{u}_n\|_{H^m_\gamma}.$$

The technical lemmas to follow will be stated and proved only for the more complicated space $\mathcal{Z}$. Analogous versions hold for $\mathcal{Y}$, where for $n = 0$ the space H^m has to be replaced by H^m_γ.

Lemma 2. *We define a linear operator* $J : \mathcal{Z} \to C^0_b(\mathbb{R}^d \times [0, 2\pi], \mathbb{C}^2)$ *by*

$$(J\tilde{u})(x,t) = U(x,t) := \sum_{n\in\mathbb{Z}} \tilde{u}_n(x) e^{int}, \quad \tilde{u} = (\tilde{u}_n)_{n\in\mathbb{Z}} \in \mathcal{Z}. \tag{22}$$

Then J *is bounded.*

Proof: Using $m > d/2$ and Sobolev's embedding theorem in d space dimensions, we have the continuous embedding $H^m(\mathbb{R}^d) \subset C^0_b(\mathbb{R}^d)$. Therefore in particular

$$|U(x,t)| \le \sum_{n\in\mathbb{Z}} |\tilde{u}_n(x)| \le \sum_{n\in\mathbb{Z}} \sup_{y\in\mathbb{R}} |\tilde{u}_n(y)| \le C \sum_{n\in\mathbb{Z}} \|\tilde{u}_n\|_{H^m} \le C\|\tilde{u}\|_{\mathcal{Z}}$$

due to $H^m_\gamma \subset H^m$, whence $\|J\tilde{u}\|_{C^0_b} \le C\|\tilde{u}\|_{\mathcal{Z}}$. $\square$

The counterpart to (component-wise) multiplication UV in physical space is given by the convolution $(\sum_{k\in\mathbb{Z}} \tilde{u}_{n-k}\tilde{v}_k)_{n\in\mathbb{Z}}$, since

$$U(x,t)V(x,t) = \sum_{l\in\mathbb{Z}} \tilde{u}_l(x)e^{ilt} \sum_{j\in\mathbb{Z}} \tilde{v}_j(x)e^{ijt} = \sum_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} \tilde{u}_{n-k}(x)\tilde{v}_k(x) \right) e^{int}.$$

Lemma 3. *For* $\tilde{u} = (\tilde{u}_n)_{n\in\mathbb{Z}} \in \mathcal{Z}$ *and* $\tilde{v} = (\tilde{v}_n)_{n\in\mathbb{Z}} \in \mathcal{Z}$ *we define* $\tilde{u} * \tilde{v} \in \mathcal{Z}$ *by*

$$(\tilde{u} * \tilde{v})_n = \sum_{k\in\mathbb{Z}} \tilde{u}_{n-k}\tilde{v}_k\,, \quad n \in \mathbb{Z}.$$

Then there exists $C > 0$ *such that for all* $\tilde{u}, \tilde{v} \in \mathcal{Z}$ *we have*

$$\|\tilde{u} * \tilde{v}\|_{\mathcal{Z}} \le C\|\tilde{u}\|_{\mathcal{Z}}\|\tilde{v}\|_{\mathcal{Z}}.$$

Proof: First we note that $\|uv\|_{H^m} \le C\|u\|_{H^m}\|v\|_{H^m}$ and $\|uv\|_{H^m_\gamma} \le C\|u\|_{H^m}\|v\|_{H^m_\gamma} \le C\|u\|_{H^m_\gamma}\|v\|_{H^m_\gamma}$. Therefore we obtain by definition of the norm in $\mathcal{Z}$ the estimate

$$
\begin{aligned}
\|\tilde{u} * \tilde{v}\|_{\mathcal{Z}} &= \Big\| \sum_{k\in\mathbb{Z}} \tilde{u}_{-k}\tilde{v}_k \Big\|_{H^m} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \Big\| \sum_{k\in\mathbb{Z}} \tilde{u}_{n-k}\tilde{v}_k \Big\|_{H^m_\gamma} \\
&\leq C \sum_{k\in\mathbb{Z}} \|\tilde{u}_{-k}\|_{H^m} \|\tilde{v}_k\|_{H^m} + C \sum_{n\in\mathbb{Z}\setminus\{0\}} \|\tilde{u}_n\|_{H^m_\gamma} \|\tilde{v}_0\|_{H^m} \\
&\quad + C \sum_{n\in\mathbb{Z}\setminus\{0\}} \|\tilde{u}_0\|_{H^m} \|\tilde{v}_n\|_{H^m_\gamma} \\
&\quad + C \sum_{n\in\mathbb{Z}\setminus\{0\}} \sum_{k\in\mathbb{Z}\setminus\{0,n\}} \|\tilde{u}_{n-k}\|_{H^m_\gamma} \|\tilde{v}_k\|_{H^m_\gamma} \\
&\leq C \Big(\sum_{n\in\mathbb{Z}} \|\tilde{u}_n\|_{H^m} \Big) \Big(\sum_{m\in\mathbb{Z}} \|\tilde{v}_m\|_{H^m} \Big) + C\|\tilde{u}\|_{\mathcal{Z}} \|\tilde{v}_0\|_{H^m} \\
&\quad + C\|\tilde{u}_0\|_{H^m} \|\tilde{v}\|_{\mathcal{Z}} + C \Big(\sum_{n\in\mathbb{Z}\setminus\{0\}} \|\tilde{u}_n\|_{H^m_\gamma} \Big) \Big(\sum_{m\in\mathbb{Z}\setminus\{0\}} \|\tilde{v}_m\|_{H^m_\gamma} \Big) \\
&\leq C\|\tilde{u}\|_{\mathcal{Z}} \|\tilde{v}\|_{\mathcal{Z}} ,
\end{aligned}
$$

as was to be shown. □

The previous lemma leads to bounds on the nonlinear terms N and F in (20).

Lemma 4. *Define $N : \mathcal{Z} \to \mathcal{Z}$ by $N(\tilde{u})_n = N_n(J\tilde{u})$ and $F : \mathcal{Z} \times \mathcal{Y} \to \mathcal{Y}$ by $F(\tilde{u}, \tilde{w})_n = F_n(J\tilde{u}, J\tilde{w})$ for $\tilde{u} \in \mathcal{Z}$ and $\tilde{w} \in \mathcal{Y}$, cf. Lemma 2 and (21). Then there exists $C > 0$ such that*

$$
\|N(\tilde{u})\|_{\mathcal{Z}} \leq C\|\tilde{u}\|_{\mathcal{Z}}^2 \quad \textit{and} \quad \|F(\tilde{u}, \tilde{w})\|_{\mathcal{Y}} \leq C\|\tilde{u}\|_{\mathcal{Z}} \|\tilde{w}\|_{\mathcal{Y}} \tag{23}
$$

for $\tilde{u} \in \mathcal{Z}$ with $\|\tilde{u}\|_{\mathcal{Z}} \leq 1$ and $\tilde{w} \in \mathcal{Y}$ with $\|\tilde{w}\|_{\mathcal{Y}} \leq 1$. Moreover, there is $C > 0$ such that

$$
\|N(\tilde{u}) - N(\tilde{v})\|_{\mathcal{Z}} \leq C\Big(\|\tilde{u}\|_{\mathcal{Z}} + \|\tilde{v}\|_{\mathcal{Z}} \Big) \|\tilde{u} - \tilde{v}\|_{\mathcal{Z}} \quad \textit{and} \tag{24}
$$

$$
\begin{aligned}
\|F(\tilde{u}, \tilde{w}) - F(\tilde{v}, \tilde{z})\|_{\mathcal{Y}} \leq C\Big(&\|\tilde{u}\|_{\mathcal{Z}} + \|\tilde{v}\|_{\mathcal{Z}} + \|\tilde{w}\|_{\mathcal{Y}} + \|\tilde{z}\|_{\mathcal{Y}} \Big) \\
&\times \Big(\|\tilde{u} - \tilde{v}\|_{\mathcal{Z}} + \|\tilde{w} - \tilde{z}\|_{\mathcal{Y}} \Big)
\end{aligned} \tag{25}
$$

for $\tilde{u}, \tilde{v} \in \mathcal{Z}$ with $\|\tilde{u}\|_{\mathcal{Z}}, \|\tilde{v}\|_{\mathcal{Z}} \leq 1$ and $\tilde{w}, \tilde{z} \in \mathcal{Y}$ with $\|\tilde{w}\|_{\mathcal{Y}}, \|\tilde{z}\|_{\mathcal{Y}} \leq 1$.

Proof : We have

$$
\|N(\tilde{u})\|_{\mathcal{Z}} = \|N_0(J\tilde{u})\|_{H^m} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \|N_n(J\tilde{u})\|_{H^m_\gamma} , \tag{26}
$$

and a contribution $U^k = (J\tilde{u})^k$ to $N(U) = N(J\tilde{u})$ leads to the n'th coefficient $(\tilde{u} * \ldots * \tilde{u})_n$, with a k-fold convolution. Hence this gives the contribution

$$
\|(\tilde{u} * \ldots * \tilde{u})_0\|_{H^m} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \|(\tilde{u} * \ldots * \tilde{u})_n\|_{H^m_\gamma} = \|\tilde{u} * \ldots * \tilde{u}\|_{\mathcal{Z}} \leq C\|\tilde{u}\|_{\mathcal{Z}}^k
$$

to the right-hand side of (26), where we have used Lemma 3 in the last step. Since $k \geq 2$ for such k by (H2), $\|\tilde{u}\|_{\mathcal{Z}} \leq 1$ implies $\|\tilde{u}\|_{\mathcal{Z}}^k \leq \|\tilde{u}\|_{\mathcal{Z}}^2$. Since the estimates on the remainder term in the Taylor polynomial are similar we have $\|N(\tilde{u})\|_{\mathcal{Z}} \leq C\|\tilde{u}\|_{\mathcal{Z}}^2$, and the proofs of the estimates on F in (23) and on the differences in (24) and (25) are analogous. □

We also need to bound the part in (20) which contains $R_\alpha(x)$.

Lemma 5. *There exists $C > 0$ such that*

$$\|e^{-\beta x_1} R_\alpha(x) W\|_{H^m_\gamma} \leq C\|W\|_{H^m_\gamma}, \quad W \in H^m_\gamma.$$

Proof: This is a direct consequence of the exponential decay of $R_\alpha(x)$ and its derivatives up to order m, since (H1) implies $\|e^{-\beta x_1} R_\alpha(x)\|_{C^m_b} \leq C$, hence

$$\begin{aligned}\|e^{-\beta x_1} R_\alpha(x) W\|_{H^m_\gamma} &= \|(e^{-\beta x_1} R_\alpha(x))(W e^{\gamma\sqrt{1+|x|^2}})\|_{H^m} \\ &\leq C\|e^{-\beta x_1} R_\alpha(x)\|_{C^m_b} \|W e^{\gamma\sqrt{1+|x|^2}}\|_{H^m} \leq C\|W\|_{H^m_\gamma}\end{aligned}$$

as desired. □

The next lemma is about linear operators in $\mathcal{Z}$.

Lemma 6. *Let a linear operator $L : \mathcal{Z} \to \mathcal{Z}$ be defined component-wise as $(L\tilde{u})_n = L_n \tilde{u}_n$ for $\tilde{u} = (\tilde{u}_n)_{n\in\mathbb{Z}} \in \mathcal{Z}$. Then we have*

$$\|L\tilde{u}\|_{\mathcal{Z}} \leq \Big(\|L_0\|_{H^m \to H^m} + \sup_{n\in\mathbb{Z}\setminus\{0\}} \|L_n\|_{H^m_\gamma \to H^m_\gamma}\Big)\|\tilde{u}\|_{\mathcal{Z}} .$$

Proof: We obtain

$$\begin{aligned}\|L\tilde{u}\|_{\mathcal{Z}} &= \|L_0\tilde{u}_0\|_{H^m} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \|L_n\tilde{u}_n\|_{H^m_\gamma} \\ &\leq \|L_0\|_{H^m\to H^m}\|\tilde{u}_0\|_{H^m} \\ &\quad + \Big(\sup_{n\in\mathbb{Z}\setminus\{0\}} \|L_n\|_{H^m_\gamma\to H^m_\gamma}\Big) \sum_{n\in\mathbb{Z}\setminus\{0\}} \|\tilde{u}_n\|_{H^m_\gamma} ,\end{aligned}$$

whence we can use $\|\tilde{u}_0\|_{H^m} \leq \|\tilde{u}\|_{\mathcal{Z}}$ as well as $\sum_{n\in\mathbb{Z}\setminus\{0\}} \|\tilde{u}_n\|_{H^m_\gamma} \leq \|\tilde{u}\|_{\mathcal{Z}}$ to find the desired estimate. □

By assumption (H3) the operator Λ_α possesses two purely imaginary eigenvalues $\lambda_0^\pm(\alpha)$, whereas the rest of the spectrum is bounded away from the imaginary axis at distance $\mathcal{O}(\min\{c^2/4, \mu\}) = \mathcal{O}(1)$. Thus there exist Λ_α-invariant projections $P^\pm_{\alpha,c}$ onto the "center" subspace spanned by the associated normalized eigenfunctions φ^+_α and φ^-_α, given by

$$P^+_{\alpha,c} u = (\varphi^{+,*}_\alpha, u)_{L^2}\, \varphi^+_\alpha \quad \text{and} \quad P^-_{\alpha,c} u = (\varphi^{-,*}_\alpha, u)_{L^2}\, \varphi^-_\alpha. \tag{27}$$

Here $\varphi^{\pm,*}_\alpha$ denote the associated normalized eigenfunctions of the adjoint operator Λ^*_α. The projection on the bounded "stable" part is $P^\pm_{\alpha,s} = I - P^\pm_{\alpha,c}$. By construction we have $P^\pm_{\alpha,c}\Lambda_\alpha = \Lambda_\alpha P^\pm_{\alpha,c}$ and $P^\pm_{\alpha,s}\Lambda_\alpha = \Lambda_\alpha P^\pm_{\alpha,s}$. With the help of these projections we split $W_{\pm 1}$ in (20) as $W_1 = W_{1c} + W_{1s}$ and $W_{-1} = W_{-1c} + W_{-1s}$, where $W_{\pm 1c} = P^{\pm}_{\alpha,c} W_1$ as well as $W_{\pm 1s} = P^{\pm}_{\alpha,s} W_1$. Applying this decomposition to (20) we obtain

$$\left.\begin{aligned} U_n &= -(B - in\omega I)^{-1}\Big(e^{-\beta x_1} R_\alpha(x) W_n + N_n(U)\Big), \qquad (n \neq 0),\\ W_n &= -(\Lambda_\alpha - in\omega I)^{-1} F_n(U,W), \qquad (n \neq \pm 1),\\ W_{\pm 1s} &= -(\Lambda_\alpha \mp i\omega I)^{-1} P^{\pm}_{\alpha,s} F_{\pm 1}(U,W),\\ U_0 &= -B^{-1}\Big(e^{-\beta x_1} R_\alpha(x) W_0 + N_0(U)\Big),\\ \pm i\omega W_{\pm 1c} &= \Lambda_\alpha W_{\pm 1c} + P^{\pm}_{\alpha,c} F_{\pm 1}(U,W). \end{aligned}\right\} \tag{28}$$

By the arguments indicated above, it will turn out that the first three equations are well defined and allow us to compute U_n for $n \neq 0$, W_n for $n \neq \pm 1$, and $W_{\pm 1s}$ in terms of the $W_{\pm 1c}$ and of U_0. The following lemma provides the corresponding technical framework for this step.

Lemma 7. *There exists $C > 0$ such that for ω close enough to ω_0 the following estimates hold.*

$$\begin{aligned} \|(B - in\omega I)^{-1}\|_{H^m_\gamma \to H^m_\gamma} &\leq C, \qquad (n \in \mathbb{Z} \setminus \{0\}),\\ \|(\Lambda_\alpha - in\omega I)^{-1}\|_{H^m_\gamma \to H^m_\gamma} &\leq C, \qquad (n \in \mathbb{Z} \setminus \{-1,1\}),\\ \|(\Lambda_\alpha \mp i\omega I)^{-1} P^{\pm}_{\alpha,s}\|_{H^m_\gamma \to H^m_\gamma} &\leq C. \end{aligned}$$

Proof: Concerning the first bound, we observe that the solution u of the equation $(B - in\omega I)u = f$ is given by

$$\hat{u}(\xi) = -(|\xi|^2 + ic\xi_1 + in\omega)^{-1} \hat{f}(\xi)$$

for $\xi \in \mathbb{R}^d$, recall (2). Since $||\xi|^2 + i(c\xi_1 + n\omega)|^2 = |\xi|^4 + (c\xi_1 + n\omega)^2 \geq (\omega/2)^2 \chi_{\{|\xi| \leq \omega/2c\}} + \delta^2 (1 + |\xi|^2)^2 \chi_{\{|\xi| \geq \omega/2c\}}$ with $\delta = \omega^2/(\omega^2 + 4c^2)$, it follows for $f \in H^m$ that $\hat{u} \in L^2_{m+2}$, thus $u \in H^{m+2}$. Then the estimate on $\|(B - in\omega I)^{-1}\|_{H^m_\gamma \to H^m_\gamma}$ is obtained as follows. Let $f \in H^m_\gamma \subset H^m$ be given, whence $g = f e^{\gamma \sqrt{1+|x|^2}} \in H^m$. With the solution u of $(B - in\omega I)u = f$ we introduce $v(x) = u(x) e^{\gamma \sqrt{1+|x|^2}}$. A short calculation reveals that v satisfies

$$(B - in\omega I)v + \gamma L v = g,$$

where

$$(Lv)(x) = -2\, \frac{x \cdot \nabla v(x)}{\sqrt{1+|x|^2}} + \left(\gamma\, \frac{|x|^2}{1+|x|^2} - \frac{(d-1)|x|^2 + d}{(1+|x|^2)^{3/2}} - c\, \frac{x_1}{\sqrt{1+|x|^2}}\right) v(x).$$

Since $L : H^{m+2} \to H^m$ is bounded, using a Neumann series we see that $(B - in\omega I) + \gamma L : H^{m+2} \to H^m$ is invertible with a bounded inverse, provided that $\gamma > 0$ is chosen sufficiently small. Hence $v \in H^{m+2}$, i.e., $u \in H^{m+2}_\gamma$, and moreover $\|u\|_{H^{m+2}_\gamma} = \|v\|_{H^{m+2}} \leq C\|g\|_{H^m} = C\|f\|_{H^m_\gamma}$.

Therefore even $(B - in\omega I)^{-1} : H^m_\gamma \to H^{m+2}_\gamma$ is bounded, but in the sequel we will only need that this operator is bounded $H^m_\gamma \to H^m_\gamma$.

So the spectrum of B in H^m_γ keeps well separated from $in\omega$ for $n \neq 0$ and γ sufficiently small. The other two estimates follow in a more abstract but similar way, as Λ_α is a sectorial operator in H^m (cf. [10]). Again the spectrum of Λ_α in H^m_γ keeps well separated from $in\omega$, if $\gamma > 0$ is assumed to be small. □

Returning to (28), it is not obvious that the fourth equation

$$U_0 = -B^{-1}\Big(e^{-\beta x_1} R_\alpha(x) W_0 + N_0(U)\Big) \tag{29}$$

for U_0 is well defined, due to the continuous spectrum of B up to the imaginary axis. With the next lemma we bound the operator B^{-1} from $H^{m-2} \cap L^1$ to H^m, as is possible in four or more space dimensions.

Lemma 8. *Suppose that $d \geq 4$. Then for all $f \in H^{m-2} \cap L^1$ the equation*

$$Bu = \Delta u + c\partial_{x_1} u = f$$

has a unique solution $u = B^{-1} f \in H^m$ which satisfies

$$\|u\|_{H^m} \leq C \|f\|_{H^{m-2} \cap L^1}. \tag{30}$$

Proof: From $f \in H^{m-2} \cap L^1$ we obtain for the Fourier transform that $\hat{f} \in L^2_{m-2} \cap C^0_b$. With a smooth cut-off function χ taking its values in $[0, 1]$ such that $\chi(\xi) = 1$ for $|\xi| \leq 1$ and $\chi(\xi) = 0$ for $|\xi| \geq 2$ we define $\hat{f}_1 = \hat{f}\chi$ and $\hat{f}_2 = \hat{f}(1 - \chi)$, so that $\hat{f} = \hat{f}_1 + \hat{f}_2$. If we further introduce u_1 and u_2 by

$$\hat{u}_1(\xi) = -\frac{\hat{f}_1(\xi)}{|\xi|^2 + ic\xi_1} \quad \text{and} \quad \hat{u}_2(\xi) = -\frac{\hat{f}_2(\xi)}{|\xi|^2 + ic\xi_1},$$

then $u = u_1 + u_2$. Moreover,

$$\begin{aligned}
\|u_1\|^2_{H^m} = \|\hat{u}_1\|^2_{L^2_m} &= \int_{\mathbb{R}^d} \frac{|\hat{f}(\xi)|^2 \chi^2(\xi)}{||\xi|^2 + ic\xi_1|^2} (1 + |\xi|^2)^m \, d\xi \\
&\leq C \|\hat{f}\|^2_{C^0_b} \int_{|\xi| \leq 2} \frac{d\xi}{|\xi|^4 + c^2 \xi_1^2} \\
&\leq C \|f\|^2_{L^1} \int_0^2 dr \int_0^2 d\xi_1 \, \frac{r^{d-2}}{r^4 + c^2 \xi_1^2} \leq C \|f\|^2_{L^1},
\end{aligned}$$

due to $d \geq 4$. In addition,

$$\begin{aligned}
\|u_2\|^2_{H^m} &= \int_{|\xi| \geq 1} \frac{|\hat{f}(\xi)|^2 (1 - \chi(\xi))^2}{|\xi|^4 + c^2 \xi_1^2} (1 + |\xi|^2)^m \, d\xi \\
&\leq C \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^{m-2} \, d\xi \leq C \|f\|^2_{H^{m-2}},
\end{aligned}$$

whence (30) follows. □

For the proof of exchange of stability later in Section 4 we additionally need to have estimates on $\|u\|_{L^\infty}$ and $\|u\|_{L^q}$ (with some $1 < q < d/2$) in terms of f for $u = B^{-1}f$ in Lemma 8.

Lemma 9. *Let $d \geq 4$. Then there exists $q \in]1, d/2[$ such that for every $m > d/2$ there is a constant $C > 0$ such that*

$$\|u\|_{L^\infty} + \|u\|_{L^q} \leq C\|f\|_{H^m \cap L^1} \tag{31}$$

for $u = B^{-1}f$.

Proof: Concerning the L^∞-estimate, we use the notation of Lemma 8 and get

$$\|\hat{u}_1\|_{L^1} \leq C\|\hat{f}\|_{C_b^0} \int_{|\xi|\leq 2} \frac{d\xi}{|\xi|^2} \leq C\|f\|_{L^1} \int_0^2 dr\, \frac{r^{d-1}}{r^2} \leq C\|f\|_{L^1}.$$

In addition, by Hölder's inequality,

$$\begin{aligned}\|\hat{u}_2\|_{L^1} &\leq C \int_{|\xi|\geq 1} \frac{|\hat{f}(\xi)|}{|\xi|^2}\, d\xi \\ &\leq C\Big(\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 (1+|\xi|^2)^{m-2}\, d\xi\Big)^{1/2} \\ &\qquad \times \Big(\int_{|\xi|\geq 1} \frac{1}{|\xi|^4}\, (1+|\xi|^2)^{-(m-2)}\, d\xi\Big)^{1/2} \\ &\leq C\|f\|_{H^{m-2}};\end{aligned}$$

recall that $m > d/2$, whence $\int_1^\infty dr\, r^{d-1-2m} < \infty$. Consequently,

$$\|u\|_{L^\infty} \leq C\|\hat{u}\|_{L^1} \leq C\|\hat{u}_1\|_{L^1} + C\|\hat{u}_2\|_{L^1} \leq C\|f\|_{H^{m-2}\cap L^1},$$

and the right-hand side in particular is majorized by $\|f\|_{H^m \cap L^1}$.

To validate the L^q-estimate, we note that in the case $d \geq 5$ we can choose $q = 2 \in]1, d/2[$, since upon taking $m = 0$ in Lemma 8 we clearly obtain $\|u\|_{L^2} \leq C(\|f\|_{L^1} + \|f\|_{L^2}) \leq C\|f\|_{H^m \cap L^1}$. Hence it remains to deal with the case $d = 4$, where $q \in]1, 2[$ is needed. We will explain, more generally, that in fact the desired estimate holds for all $q > (d+1)/(d-1)$; since $1 < (d+1)/(d-1) < d/2$ for $d \geq 4$, this will establish the claim. Again we write

$$u = u_1 + u_2, \quad \text{with} \quad u_1 = G_1 * f \quad \text{and} \quad u_2 = G_2 * f,$$

where

$$\hat{G}_1(\xi) = -\frac{\chi(\xi)}{|\xi|^2 + ic\xi_1} \quad \text{and} \quad \hat{G}_2(\xi) = -\frac{1-\chi(\xi)}{|\xi|^2 + ic\xi_1},$$

with a cut-off function $\chi \in C_0^\infty(\mathbb{R}^d)$ such that $\chi(\xi) = 1$ for $|\xi| \leq 1$ and $\chi(\xi) = 0$ for $|\xi| \geq 2$. The Hörmander multiplier theorem (see [5, Cor. 8.11]) implies in particular that $f \mapsto G_2 * f$ is bounded on any $L^q(\mathbb{R}^d)$, $q \in]1, \infty[$. Thus $\|u_2\|_{L^q} \leq C\|f\|_{L^q}$. Furthermore, $H^m \subset L^\infty$ according to Sobolev's embedding theorem, whence by interpolation we have

$$\|f\|_{L^q} \leq \|f\|_{L^\infty}^{1-\frac{1}{q}} \|f\|_{L^1}^{\frac{1}{q}} \leq C\|f\|_{H^m}^{1-\frac{1}{q}} \|f\|_{L^1}^{\frac{1}{q}} \leq C\|f\|_{H^m \cap L^1},$$

so that $\|u_2\|_{L^q} \leq C\|f\|_{H^m \cap L^1}$.

In order to bound u_1, we use Young's inequality to get

$$\|u_1\|_{L^q} \leq \|G_1\|_{L^q}\|f\|_{L^1} \leq \|G_1\|_{L^q}\|f\|_{H^m \cap L^1}.$$

Thus it remains to derive an estimate on $\|G_1\|_{L^q}$. If we introduce G_3 by its Fourier transformation $\hat{G}_3(\xi) = -(|\xi|^2 + ci\xi_1)^{-1}e^{-|\xi|^2/2}$, then

$$\|G_1\|_{L^q} \leq \|G_1 - G_3\|_{L^q} + \|G_3\|_{L^q} \leq C + \|G_3\|_{L^q},$$

since $\hat{G}_1 - \hat{G}_3$ is a Schwartz function, hence also $G_1 - G_3$ is a Schwartz function. The inverse Fourier transform of $\hat{G}_3$ can be computed more or less explicitly, with the help of [7], using the same argument as for $G_1 - G_3$ a number of times. Note that the function $v = G_3$ solves $\Delta v + c\partial_{x_1} v = e^{-|x|^2/2}$, provided that we define the Fourier transform as

$$\hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x)\, dx.$$

It is then possible to see that $G_3 \in L^q$ for $q > (d+1)/(d-1)$. We skip this very lengthy, not instructive, computation. Instead of this calculation we give a heuristic argument which indicates where the restriction $q > (d+1)/(d-1)$ comes from. To do so, we recall that the Fourier transform is continuous $L^p \to L^q$, with $1/p + 1/q = 1$, if $p \in [1, 2]$ and accordingly $q \in [2, \infty]$. Although we need a bound on $\|u_1\|_{L^q}$ for $q < 2$ in the case $d = 4$, we ask for which $p > 1$ we can bound $\|\hat{u}_1\|_{L^p}$ by $C\|f\|_{H^m \cap L^1}$. For this it is sufficient that the integral

$$I = \int_{|\xi| \leq 2} \frac{d\xi}{||\xi|^2 + ic\xi_1|^p}$$

is finite. It can be estimated by

$$\begin{aligned} I &\leq C \int_{|\xi| \leq 2} \frac{d\xi}{|\xi|^{2p} + c^p|\xi_1|^p} \leq C \int_0^2 dr \int_0^2 d\xi_1 \frac{r^{d-2}}{r^{2p} + c^p|\xi_1|^p} \\ &\leq C \int_0^2 dr\; r^{d-2-2p+2}, \end{aligned}$$

the latter by the change of variables $\zeta = r^{-2}\xi_1$, $d\zeta = r^{-2}d\xi_1$. Hence I is finite, if $p < (d+1)/2$, or stated differently, if $q > (d+1)/(d-1)$. □

So far we have considered only the case that assumption (H2) holds for the nonlinearity. It is more or less straightforward that the above steps can be reworked if (H1), (H2) are replaced by (H1)', (H2)'. However, under the latter hypothesis we can allow for space dimensions $d \geq 3$, since in the later proof of Theorem 2 we can instead of Lemma 8 invoke the following observation.

Lemma 10. *Suppose that $d \geq 3$ and $1 \leq j \leq d$. Then for all $f \in H^{m-1} \cap L^1$ the equation*

$$Bu = \Delta u + c\partial_{x_1} u = \partial_{x_j} f$$

has a unique solution $u = B^{-1}(\partial_{x_j} f) \in H^m$ which satisfies

$$\|u\|_{H^m} \leq C\|f\|_{H^{m-1}\cap L^1}.$$

Proof: From $f \in H^{m-1} \cap L^1$ it follows for the Fourier transform that $\hat{f} \in L^2_{m-1} \cap C^0_b$. Once more we split $\hat{f} = \hat{f}_1 + \hat{f}_2$, where $\hat{f}_1 = \hat{f}\chi$ and $\hat{f}_2 = \hat{f}(1-\chi)$, and χ is as in Lemma 8. In Fourier space we define $\hat{u} = \hat{u}_1 + \hat{u}_2$, with

$$\hat{u}_1(\xi) = \frac{i\xi_j \hat{f}_1(\xi)}{|\xi|^2 + ic\xi_1} \quad \text{and} \quad \hat{u}_2(\xi) = \frac{i\xi_j \hat{f}_2(\xi)}{|\xi|^2 + ic\xi_1}.$$

Then we distinguish two cases. For $j = 1$ we have

$$\begin{aligned} \|u_1\|^2_{H^m} &= \int_{\mathbb{R}^d} \frac{\xi_1^2 |\hat{f}(\xi)|^2 \chi^2(\xi)}{|\xi|^4 + c^2\xi_1^2} (1 + |\xi|^2)^m \, d\xi \\ &\leq C\|\hat{f}\|^2_{C^0_b} \int_{|\xi|\leq 2} 1 \, d\xi \leq C\|f\|^2_{L^1}. \end{aligned}$$

On the other hand, for $j \neq 1$ we get

$$\begin{aligned} \|u_1\|^2_{H^m} &= \int_{\mathbb{R}^d} \frac{\xi_j^2 |\hat{f}(\xi)|^2 \chi^2(\xi)}{|\xi|^4 + c^2\xi_1^2} (1 + |\xi|^2)^m \, d\xi \\ &\leq C\|\hat{f}\|^2_{C^0_b} \int_{|\xi|\leq 2} \frac{\xi_j^2}{\xi_j^4 + c^2\xi_1^2} \, d\xi \\ &\leq C\|f\|^2_{L^1} \int_{-2}^{2} \int_{-2}^{2} \frac{\xi_j^2}{\xi_j^4 + c^2\xi_1^2} \, d\xi_1 d\xi_j \\ &\leq C\|f\|^2_{L^1} \int_{-2}^{2} \arctan\Big(\frac{c\xi_1}{\xi_j^2}\Big)\Big|_{\xi_1=-2}^{\xi_1=2} d\xi_j \leq C\|f\|^2_{L^1}. \end{aligned}$$

Furthermore, for any $1 \leq j \leq d$ we see that

$$\begin{aligned} \|u_2\|^2_{H^m} &= \int_{|\xi|\geq 1} \frac{\xi_j^2 |\hat{f}(\xi)|^2 (1 - \chi(\xi))^2}{|\xi|^4 + c^2\xi_1^2} (1 + |\xi|^2)^m \, d\xi \\ &\leq C \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^{m-1} \, d\xi \leq C\|f\|^2_{H^{m-1}}, \end{aligned}$$

and hence the claim is obtained. □

It can also be verified that an additional estimate analogously to (31) from Lemma 9 holds. Therefore we will be able to solve (29) for U_0 under the assumption (H2) by means of Lemma 8, and moreover we can use Lemma 10 if (H2)' holds. We also want to remark that, provided that (H4) in Theorem 2 is satisfied, we necessarily have $U_0 = 0$ by Lemma 1, whence no restriction on the space dimension arises in this case.

From now on we continue to consider (H2) only. To finally make use of Lemma 8 in (29), it is necessary to bound both $e^{-\beta x_1} R_\alpha(x) W_0$ and $N_0(U)$.

Lemma 11. *We have the estimate*

$$\|e^{-\beta x_1} R_\alpha(x) W_0\|_{H^m \cap L^1} + \|N_0(U)\|_{H^m \cap L^1} \leq C\Big(\|W_0\|_{H^m_\gamma} + \|\tilde{u}\|^2_{\mathcal{Z}}\Big),$$

provided that $\tilde{u} \in \mathcal{Z}$ satisfies $\|\tilde{u}\|_{\mathcal{Z}} \leq 1$, where $U = J\tilde{u}$ and $\tilde{u} \in \mathcal{Z}$ are related by (22).

Proof: To begin with, $\|e^{-\beta x_1} R_\alpha(x) W_0\|_{H^m} \leq \|e^{-\beta x_1} R_\alpha(x)\|_{C_b^m} \|W_0\|_{H^m} \leq C\|W_0\|_{H^m} \leq C\|W_0\|_{H^m_\gamma}$ by (H1), and also $\|N_0(U)\|_{H^m} \leq \|N(\tilde{u})\|_{\mathcal{Z}} \leq C\|\tilde{u}\|^2_{\mathcal{Z}}$ for $\tilde{u} \in \mathcal{Z}$ with $\|\tilde{u}\|_{\mathcal{Z}} \leq 1$, cf. (26) and Lemma 4. In addition, $\|e^{-\beta x_1} R_\alpha(x) W_0\|_{L^1} \leq C\|W_0\|_{L^1} \leq C\|W_0\|_{L^2_\gamma} \leq C\|W_0\|_{H^m_\gamma}$. Next, if we consider a contribution U^k to $N(U)$ with $k \geq 2$, then the corresponding contribution to $N_0(U)$ is

$$(\tilde{u} * \ldots * \tilde{u})_0(x) = \sum_{n_1+\ldots+n_k=0} \tilde{u}_{n_1}(x) \cdot \ldots \cdot \tilde{u}_{n_k}(x).$$

If in this sum $n_j \neq 0$ for at least one $1 \leq j \leq k$, then $\tilde{u} \in \mathcal{Z}$ implies $\tilde{u}_{n_j} \in H^m_\gamma \subset L^1$. Hence $\tilde{u}_{n_l} \in H^m \subset C_b^0$ for $l \neq j$ yields $\|\tilde{u}_{n_1} \cdot \ldots \cdot \tilde{u}_{n_k}\|_{L^1} \leq \Big(\prod_{l\neq j} \|\tilde{u}_{n_l}\|_{H^m}\Big)\|\tilde{u}_{n_j}\|_{H^m_\gamma}$, and it follows that

$$\|(\tilde{u} * \ldots * \tilde{u})_0\|_{L^1} \leq C\|\tilde{u}\|^k_{\mathcal{Z}} + \|\tilde{u}_0^k\|_{L^1}.$$

To bound the last term we can use

$$\|\tilde{u}_0^k\|_{L^1} \leq \|\tilde{u}_0\|^{k-2}_{C_b^0} \|\tilde{u}_0^2\|_{L^1} \leq C\|\tilde{u}_0\|^{k-2}_{H^m} \|\tilde{u}_0\|^2_{L^2} \leq C\|\tilde{u}\|^k_{\mathcal{Z}},$$

and due to $\|\tilde{u}\|_{\mathcal{Z}} \leq 1$ and $k \geq 2$ the claim is obtained. □

Thanks to this preparation we are now in the position to solve, in terms of the $W_{\pm 1c}$, the first four equations from (28), i.e.,

$$\left.\begin{aligned} U_n &= -(B - in\omega I)^{-1}\Big(e^{-\beta x_1} R_\alpha(x) W_n + N_n(U)\Big), \qquad (n \neq 0), \\ W_n &= -(\varLambda_\alpha - in\omega I)^{-1} F_n(U, W), \qquad (n \neq \pm 1), \\ W_{\pm 1s} &= -(\varLambda_\alpha \mp i\omega I)^{-1} P^{\pm}_{\alpha,s} F_{\pm 1}(U, W), \\ U_0 &= -B^{-1}\Big(e^{-\beta x_1} R_\alpha(x) W_0 + N_0(U)\Big), \end{aligned}\right\} \tag{32}$$

with respect to the U_n for $n \in \mathbb{Z}$, W_n for $n \neq \pm 1$, and $W_{\pm 1s}$. This implicit solution will thereafter be inserted in the fifth equation from (28), which then yields a two-dimensional reduced system.

Lemma 12. *Suppose that (H2) holds and $d \geq 4$. Then there exist $\delta_1, \delta_2 > 0$ such that for all $\omega > 0$ with $|\omega - \omega_0| \leq \delta_1$ and all $W_{\pm 1c} \in H^m_\gamma$ with $\|W_{\pm 1c}\|_{H^m_\gamma} \leq \delta_2$ the system (32) has a unique solution $(\tilde{u}, \tilde{w}) = \Phi(W_c) \in \mathcal{Z} \times \mathcal{Y}$, where $W_c = (W_{-1c}, W_{1c})$ and $\tilde{u} = (\dots, U_{-2}, U_{-1}, U_0, U_1, U_2, \dots)$ as well as $\tilde{w} = (\dots, W_{-2}, W_{-1c} + W_{-1s}, W_0, W_{1c} + W_{1s}, W_2, \dots)$. In addition, $\Phi(0) = (0, 0)$, and there exists $C > 0$ such that*

$$\|\tilde{u}\|_{\mathcal{Z}} + \|\tilde{w} - W_c\|_{\mathcal{Y}} \leq C\|W_c\|_{H^m_\gamma}, \tag{33}$$

with $\tilde{w} - W_c := \tilde{w} - (\dots, 0, W_{-1c}, 0, W_{1c}, 0, \dots)$.

Proof: We fix $\omega > 0$ so close to ω_0 that Lemma 7 applies; this defines δ_1. For given $W_{\pm 1c} \in H^m_\gamma$ with $\|W_{\pm 1c}\|_{H^m_\gamma} \leq \delta_2$ sufficiently small, we have to determine

$$\begin{aligned} \tilde{u}^* &= (\dots, U_{-2}, U_{-1}, U_0, U_1, U_2, \dots) \quad \text{and} \\ \tilde{w}^* &= (\dots, W_{-2}, W_{-1s}, W_0, W_{1s}, W_2, \dots) \end{aligned}$$

such that the vectors $\tilde{u} = \tilde{u}^*$ and

$$\tilde{w} = \tilde{w}^* + W_c = \tilde{w}^* + (\dots, 0, W_{-1c}, 0, W_{1c}, 0, \dots)$$

are the solution of (32). Therefore we fix $W_{\pm 1c} \in H^m_\gamma$ and define the operator

$$\begin{aligned} \mathcal{F}: \quad (\tilde{u}^*, \tilde{w}^*) &\mapsto (\tilde{u}, \tilde{w}) = \Big(\tilde{u}^*, \tilde{w}^* + (\dots, 0, W_{-1c}, 0, W_{1c}, 0, \dots)\Big) \\ &\mapsto (U, W) \mapsto (\tilde{u}^{**}, \tilde{w}^{**}) = \text{right-hand side of (32)}, \end{aligned} \tag{34}$$

where $U = J\tilde{u}$ and $W = J\tilde{w}$ are as in Lemma 2. In order to verify that $\mathcal{F}$ is a self-map of a sufficiently small ball in $\mathcal{Z} \times \mathcal{Y}$, we first use Lemma 6 in $\mathcal{Y}$, Lemma 7, and Lemma 4 to obtain

$$\begin{aligned} \|\tilde{w}^{**}\|_{\mathcal{Y}} \leq C \sup &\Big\{ \|(\Lambda_\alpha \mp i\omega I)^{-1} P^{\pm}_{\alpha,s}\|_{H^m_\gamma \to H^m_\gamma}, \\ &\qquad \|(\Lambda_\alpha - in\omega I)^{-1}\|_{H^m_\gamma \to H^m_\gamma} : n \in \mathbb{Z} \setminus \{-1, 1\} \Big\} \\ &\times \|(F_n(U, W))_{n \in \mathbb{Z}}\|_{\mathcal{Y}} \\ &\leq C\|F(\tilde{u}, \tilde{w})\|_{\mathcal{Y}} \leq C\|\tilde{u}\|_{\mathcal{Z}} \|\tilde{w}\|_{\mathcal{Y}} \\ &\leq C\|\tilde{u}^*\|_{\mathcal{Z}} \Big(\|\tilde{w}^*\|_{\mathcal{Y}} + \|W_{1c}\|_{H^m_\gamma} + \|W_{-1c}\|_{H^m_\gamma} \Big) \qquad (35) \\ &\leq C_1 \|\tilde{u}^*\|_{\mathcal{Z}} \Big(\|\tilde{w}^*\|_{\mathcal{Y}} + \delta_2 \Big), \qquad (36) \end{aligned}$$

provided that $\|\tilde{u}^*\|_{\mathcal{Z}} \le 1$ and $\|\tilde{w}^*\|_{\mathcal{Y}} \le 1 - 2\delta_2$. Analogously, it follows with Lemma 7, Lemma 8, Lemma 5, and Lemma 11 that

$$\begin{aligned}
&\|\tilde{u}^{**}\|_{\mathcal{Z}} \\
&\le C\Bigg(\sum_{n\in\mathbb{Z}\setminus\{0\}} \|e^{-\beta x_1} R_\alpha(x) W_n + N_n(U)\|_{H^m_\gamma} \\
&\qquad\qquad + \|e^{-\beta x_1} R_\alpha(x) W_0 + N_0(U)\|_{H^{m-2}\cap L^1} \Bigg) \\
&\le C\Bigg(\sum_{n\in\mathbb{Z}\setminus\{0\}} \|W_n\|_{H^m_\gamma} + \|N(\tilde{u})\|_{\mathcal{Z}} + \|W_0\|_{H^m_\gamma} + \|\tilde{u}\|^2_{\mathcal{Z}} \Bigg) \\
&\le C\Big(\|\tilde{w}\|_{\mathcal{Y}} + \|\tilde{u}\|^2_{\mathcal{Z}} \Big) \le C_1\Big(\|\tilde{w}^*\|_{\mathcal{Y}} + \delta_2 + \|\tilde{u}^*\|^2_{\mathcal{Z}} \Big), \qquad (37)
\end{aligned}$$

once more under the conditions $\|\tilde{u}^*\|_{\mathcal{Z}} \le 1$ and $\|\tilde{w}^*\|_{\mathcal{Y}} \le 1 - 2\delta_2$. We now endow $\mathcal{Z} \times \mathcal{Y}$ with the norm $\|(\tilde{u}^*, \tilde{w}^*)\|^{(1)}_{\mathcal{Z}\times\mathcal{Y}} := \|\tilde{u}^*\|_{\mathcal{Z}} + A\|\tilde{w}^*\|_{\mathcal{Y}}$, where $A = 2C_1$ and w.l.o.g. $C_1 \ge 1$. If we then choose $\delta_2 \le 1/(40C_1^2)$, we deduce from (37) and (36) for $(\tilde{u}^*, \tilde{w}^*) \in \mathcal{Z} \times \mathcal{Y}$ with $\|(\tilde{u}^*, \tilde{w}^*)\|^{(1)}_{\mathcal{Z}\times\mathcal{Y}} \le 1/(8C_1)$ the estimate

$$\begin{aligned}
&\|\mathcal{F}(\tilde{u}^*, \tilde{w}^*)\|^{(1)}_{\mathcal{Z}\times\mathcal{Y}} \\
&\quad = \|\tilde{u}^{**}\|_{\mathcal{Z}} + A\|\tilde{w}^{**}\|_{\mathcal{Y}} \\
&\quad \le C_1\Big(\|\tilde{w}^*\|_{\mathcal{Y}} + \delta_2 + \|\tilde{u}^*\|^2_{\mathcal{Z}} + A\|\tilde{u}^*\|_{\mathcal{Z}}\|\tilde{w}^*\|_{\mathcal{Y}} + A\delta_2\|\tilde{u}^*\|_{\mathcal{Z}} \Big) \\
&\quad \le C_1\Big(A^{-1}\|(\tilde{u}^*, \tilde{w}^*)\|_{\mathcal{Z}\times\mathcal{Y}} + \delta_2 + 2\|(\tilde{u}^*, \tilde{w}^*)\|^2_{\mathcal{Z}\times\mathcal{Y}} + A\delta_2\|(\tilde{u}^*, \tilde{w}^*)\|_{\mathcal{Z}\times\mathcal{Y}} \Big) \\
&\quad \le C_1\left(\frac{1}{16C_1^2} + \frac{1}{40C_1^2} + \frac{1}{32C_1^2} + \frac{1}{160C_1^2} \right) = \frac{1}{8C_1}.
\end{aligned}$$

Thus $\mathcal{F}$ maps the $\|\cdot\|^{(1)}_{\mathcal{Z}\times\mathcal{Y}}$-ball of radius $r = 1/(8C_1)$ into itself. Using (24) and (25), it can be verified in a similar way that $\mathcal{F}$ is a contraction, if we possibly decrease the ball further. Hence we obtain a unique fixed point $(\tilde{u}^*, \tilde{w}^*) \in \mathcal{Z} \times \mathcal{Y}$ of $\mathcal{F}$, which yields the desired solution $(\tilde{u}, \tilde{w}) = (\tilde{u}^*, \tilde{w}^* + W_c)$ of (32), cf. (34). In particular, if $W_{\pm 1c} = 0$, then $\tilde{u} = \tilde{w} = 0$ is the solution of (32), thus $\Phi(0) = (0, 0)$. Concerning (33), we first note that $(\tilde{u}^*, \tilde{w}^*) = \mathcal{F}(\tilde{u}^*, \tilde{w}^*) = (\tilde{u}^{**}, \tilde{w}^{**})$ together with (35) yields

$$\|\tilde{w} - W_c\|_{\mathcal{Y}} = \|\tilde{w}^*\|_{\mathcal{Y}} = \|\tilde{w}^{**}\|_{\mathcal{Y}} \le C\|\tilde{u}^*\|_{\mathcal{Z}}\Big(\|\tilde{w}^*\|_{\mathcal{Y}} + \|W_{1c}\|_{H^m_\gamma} \Big),$$

whence $\|\tilde{w}^*\|_{\mathcal{Y}} \le C\|\tilde{u}^*\|_{\mathcal{Z}}\|W_{1c}\|_{H^m_\gamma} \le C\|W_{1c}\|_{H^m_\gamma}$, where we have to decrease r again if necessary. From (37) we see that $\|\tilde{u}^*\|_{\mathcal{Z}} \le C(\|\tilde{w}^*\|_{\mathcal{Y}} + \|W_{1c}\|_{H^m_\gamma} + \|\tilde{u}^*\|^2_{\mathcal{Z}})$. Thus for $\|\tilde{u}^*\|_{\mathcal{Z}}$ small enough we obtain

$$\|\tilde{u}\|_{\mathcal{Z}} = \|\tilde{u}^*\|_{\mathcal{Z}} \le C(\|\tilde{w}^*\|_{\mathcal{Y}} + \|W_{1c}\|_{H^m_\gamma}) \le C\|W_{1c}\|_{H^m_\gamma}.$$

Therefore the proof of Lemma 12 is complete. □

Remark 2. The following observation will be useful later in Section 4 for the proof of exchange of stability: Given $\delta_0 > 0$, we may always arrange that

$$\|U_0\|_{L^q} + \|U_0\|_{L^\infty} \le \delta_0$$

is satisfied, with $q < d/2$ by Lemma 9. Indeed, (31), (29), Lemma 11, and (37) yield

$$\begin{aligned} \|U_0\|_{L^q} + \|U_0\|_{L^\infty} &\le C\|e^{-\beta x_1} R_\alpha(x) W_0 + N_0(U)\|_{H^m \cap L^1} \\ &\le C\Big(\|W_0\|_{H^m_\gamma} + \|\tilde{u}\|^2_{\mathcal{Z}}\Big) \\ &\le C\Big(\|\tilde{w}^*\|_{\mathcal{Y}} + \delta_2 + \|\tilde{u}^*\|^2_{\mathcal{Z}}\Big). \end{aligned} \tag{38}$$

Since $(\tilde{u}^*, \tilde{w}^*) \in \mathcal{Z} \times \mathcal{Y}$ is the unique fixed point of $\mathcal{F}$ in a $\|(\cdot,\cdot)\|^{(1)}_{\mathcal{Z}\times\mathcal{Y}}$-norm small ball, it follows that the right-hand side of (38) can be made as small as necessary by shrinking this ball and by choosing $\delta_2 > 0$ small enough in Lemma 12.

Proof of Theorem 2 : As before, we only deal with the case that (H2) is satisfied and $d \ge 4$. Then it remains to analyze the fifth equation from (28), which we restate:

$$\pm i\omega W_{\pm 1c} = \Lambda_\alpha W_{\pm 1c} + P^{\pm}_{\alpha,c} F_{\pm 1}(U, W).$$

From $W_{-1} = \overline{W_1}$ and $N(\overline{U}) = \overline{N(U)}$, cf. (H2), it follows that the "$-$"-equation is the complex conjugate of the "$+$"-equation. If we moreover express $(U, W) = (J\tilde{u}, J\tilde{w}) =: J(\tilde{u}, \tilde{w})$ by means of $(\tilde{u}, \tilde{w}) = \Phi(W_c) = \Phi(\overline{W_{1c}}, W_{1c})$ from Lemma 12, we need to find (ω, α) close to (ω_0, α_c) and a non-trivial solution $W_{1c} = W_{1c}(x)$ of

$$0 = -i\omega W_{1c} + \Lambda_\alpha W_{1c} + P^+_{\alpha,c} F_1(J\Phi(\overline{W_{1c}}, W_{1c})). \tag{39}$$

Since $W_{1c} \in \mathbb{C}\varphi^+_\alpha$, cf. (27), writing $W_{1c} = \zeta\varphi^+_\alpha$ we obtain the equation

$$0 = -i\omega\zeta\varphi^+_\alpha + \lambda^+_0(\alpha)\zeta\varphi^+_\alpha + P^+_{\alpha,c} F_1(J\Phi(\overline{\zeta\varphi^+_\alpha}, \zeta\varphi^+_\alpha))$$

to be satisfied for some $\zeta \in \mathbb{C} \setminus \{0\}$, recall $\Lambda_\alpha \varphi^+_\alpha = \lambda^+_0(\alpha)\varphi^+_\alpha$. If we now introduce $p^+_{\alpha,c}$ by $P^+_{\alpha,c} u = p^+_{\alpha,c}(u)\varphi^+_\alpha$, this can be simplified to

$$0 = -i\omega\zeta + \lambda^+_0(\alpha)\zeta + f(\alpha, \zeta), \quad f(\alpha, \zeta) := p^+_{\alpha,c}\Big(F_1(J\Phi(\overline{\zeta\varphi^+_\alpha}, \zeta\varphi^+_\alpha))\Big), \tag{40}$$

which is an equation for $\zeta \in \mathbb{C}$. Clearly $|p^+_{\alpha,c}(u)| \le C\|P^+_{\alpha,c} u\|_{H^m_\gamma} \le C\|u\|_{H^m_\gamma}$, whence (23) in Lemma 4 yields

$$\begin{aligned} \Big|p^+_{\alpha,c}\Big(F_1(J\Phi(\overline{\zeta\varphi^+_\alpha}, \zeta\varphi^+_\alpha))\Big)\Big| &\le C\|F_1(J\Phi(\overline{\zeta\varphi^+_\alpha}, \zeta\varphi^+_\alpha))\|_{H^m_\gamma} \\ &\le C\|F(J\Phi(\overline{\zeta\varphi^+_\alpha}, \zeta\varphi^+_\alpha))\|_{\mathcal{Y}} \\ &\le C\|\tilde{u}\|_{\mathcal{Z}}\|\tilde{w}\|_{\mathcal{Y}}, \end{aligned}$$

where we have employed the notation

$$(\tilde{u}, \tilde{w}) = \Phi(W_c) = \Phi(\overline{\zeta\varphi_\alpha^+}, \zeta\varphi_\alpha^+).$$

Recalling $W_c \cong (\ldots, 0, W_{-1c}, 0, W_{1c}, 0, \ldots)$ thus (33) in Lemma 12 implies

$$\begin{aligned}|f(\alpha,\zeta)| &= \Big|p_{\alpha,c}^+\Big(F_1(J\Phi(\overline{\zeta\varphi_\alpha^+}, \zeta\varphi_\alpha^+))\Big)\Big| \le C\|\tilde{u}\|_{\mathcal{Z}}\Big(\|\tilde{w} - W_c\|_{\mathcal{Y}} + \|W_c\|_{\mathcal{Y}}\Big)\\ &\le C\|W_c\|^2_{H^m_\gamma} \le C\|\zeta\varphi_\alpha^+\|^2_{H^m_\gamma} \le C|\zeta|^2.\end{aligned}$$

Therefore the nonlinear part f in (40) is of higher order at $\zeta = 0$, and a similar reasoning shows that even

$$|f(\alpha,\zeta)| \le C|\zeta|^p$$

holds, with the p from hypothesis (H2). Analogously to the case of classical Hopf bifurcation, cf. [16], the implicit function theorem can be applied to (40), as soon as the zero solution is divided out; it will also be sufficient to find real-valued solutions $r = \zeta \in \mathbb{R}$. Hence we consider the complex-valued smooth function

$$\Gamma(r;\rho,\varepsilon) = \begin{cases} -i(\omega_0+\rho) + \lambda_0^+(\alpha_c+\varepsilon) + r^{-1}f(\alpha_c+\varepsilon, r) & : \quad r \neq 0 \\ -i(\omega_0+\rho) + \lambda_0^+(\alpha_c+\varepsilon) & : \quad r = 0 \end{cases}$$

and note that $\Gamma(0;0,0) = 0$ due to $\lambda_0^+(\alpha_c) = i\omega_0$, cf. (4) in (H3). Moreover, the Jacobi matrix

$$D_{\rho,\varepsilon}\Gamma(r;\rho,\varepsilon)|_{r=\rho=\varepsilon=0} = \begin{pmatrix} 0 & \frac{d}{d\alpha}\operatorname{Re}\lambda_0^+(\alpha)|_{\alpha=\alpha_c} \\ -1 & \frac{d}{d\alpha}\operatorname{Im}\lambda_0^+(\alpha)|_{\alpha=\alpha_c} \end{pmatrix}$$

w.r. to ρ, ε has

$$\det D_{\rho,\varepsilon}\Gamma(r;\rho,\varepsilon)|_{r=\rho=\varepsilon=0} = \frac{d}{d\alpha}\operatorname{Re}\lambda_0^+(\alpha)|_{\alpha=\alpha_c} > 0$$

by (5) in (H3). Hence we find functions $r \mapsto (\rho(r), \varepsilon(r))$ with $\rho(0) = \varepsilon(0) = 0$ such that $-i(\omega_0 + \rho(r)) + \lambda_0^+(\alpha_c + \varepsilon(r)) + r^{-1}f(\alpha_c + \varepsilon(r), r) = 0$ for $|r|$ small enough. Differentiating this relation it follows that, depending on the degree of the nonlinearity, $\varepsilon^{(k)}(0) \neq 0$ for some first k. Thus the function $r \mapsto \varepsilon(r)$ can locally be inverted to yield a function $\varepsilon \mapsto r(\varepsilon)$. This results in $-i(\omega_0+\rho_1(\varepsilon))r(\varepsilon)+\lambda_0^+(\alpha_c+\varepsilon)r(\varepsilon)+f(\alpha_c+\varepsilon, r(\varepsilon)) = 0$ for ε sufficiently small, where $\rho_1(\varepsilon) = \rho(r(\varepsilon))$. Setting $\omega = \omega_0 + \rho_1(\varepsilon)$, $\alpha = \alpha_c + \varepsilon$, and $W_{1c}(x) = r(\varepsilon)\varphi^+_{\alpha_c+\varepsilon}(x)$, we thus have constructed the desired solutions of (39). Working the foregoing steps backward, we finally obtain a $2\pi/\omega$-time-periodic solution $U = U(x,t)$ of (18) satisfying $W(x,t) = e^{\beta x_1}U(x,t)$. This completes the proof of Theorem 2. □

Remark 3. In the generic case of $\kappa_3 \neq 0$ one has $\underline{f(\alpha, r)} = (\kappa_3 + i\kappa_4)r^3 + \mathcal{O}(r^4)$, where $\kappa_3 + i\kappa_4 = \lim_{r\to 0} r^{-3}(\varphi^{+,*}_{\alpha_c}, F_1(J\Phi(r\overline{\varphi^{+}_{\alpha_c}}, r\varphi^{+}_{\alpha_c})))_{L^2}$ in case of the nonlinearity (6) could in principle be calculated explicitly. Differentiating $\Gamma(r; \rho(r), \varepsilon(r)) = 0$ then yields $\rho'(0) = \varepsilon'(0) = 0$, and differentiating this relation a second time we see that

$$0 = -i\rho''(0) + \frac{d}{d\alpha}\,\lambda_0^+(\alpha)|_{\alpha=\alpha_c}\varepsilon''(0) + 2(\kappa_3 + i\kappa_4).$$

Writing $\frac{d}{d\alpha}\,\lambda_0^+(\alpha)|_{\alpha=\alpha_c} = \kappa_1 + i\kappa_2$ with $\kappa_1 > 0$, this can be stated differently as $\varepsilon''(0) = -2\kappa_3/\kappa_1$ and $\rho''(0) = 2(\kappa_1\kappa_4 - \kappa_2\kappa_3)/\kappa_1$. Hence if $\kappa_3 \neq 0$, then $\varepsilon(r) = -(\kappa_3/\kappa_1)r^2 + \mathcal{O}(r^3)$ yields $\alpha - \alpha_c = \varepsilon = -\mathrm{sgn}(\kappa_3)\tilde{\varepsilon}$, where $\tilde{\varepsilon} = |\varepsilon| > 0$ is the ε which appears in the formulation of Theorem 2. Depending on the sign of κ_3, we thus obtain a sub- or supercritical Hopf bifurcation.

Note that there are no quadratic terms in the reduced system due to the fact that $e^{\pm it}e^{\pm it} \notin \{e^{-it}, e^{it}\}$, i.e., the quadratic interaction of critical modes always leads to noncritical modes.

4. Exchange of stability

So far we have established the stability of the trivial solution for $\alpha < \alpha_c$ and that a Hopf bifurcation occurs at $\alpha = \alpha_c$. We now turn to the issue of the stability of the bifurcating small periodic solutions, i.e. we are going to prove Theorem 3. We start again with the original equation

$$\partial_t U = BU + R_\alpha(x)U + N(U),$$

and we introduce the deviation $V(x,t) = U(x,t) - U_{\mathrm{per}}(x,t)$ of a general solution U from the time-periodic solution U_{per}, the latter given by Theorem 2. Hence we obtain

$$\partial_t V = BV + R_\alpha(x)V + N(U_{\mathrm{per}} + V) - N(U_{\mathrm{per}}) \tag{41}$$

as the equation for V. Again we need to make use of the exponentially weighted function $W(x,t) = e^{\beta x_1}V(x,t)$ with $\beta \in]0, \min\{\gamma, c/2\}[$ suitable chosen below, and $\gamma > 0$ the exponential decay rate occurring in the representation of the time periodic solutions from the last section. Note that now β is chosen different than in previous sections, where we had fixed $\beta = c/2$.

The function W then satisfies

$$\partial_t W = \Lambda_\alpha W + G(V, W), \tag{42}$$

with

$$G(V, W) = e^{\beta x_1}(N(U_{\mathrm{per}} + V) - N(U_{\mathrm{per}})).$$

Everything which has been said about the nonlinearity F around (7) applies to G, too.

Example 2. Writing $U_{\rm per} = (U_{p1}, U_{p2})$ and $V = (V_1, V_2)$, we find for the nonlinearity from (6) the expression

$$\begin{aligned} &N(U_{\rm per}+V) - N(U_{\rm per}) \\ &\quad = \begin{pmatrix} -3U_{p1}^2V_1 - 3U_{p1}V_1^2 - 2U_{p2}V_2(U_{p1}+V_1) \\ -U_{p1}V_2^2 - U_{p2}^2V_1 - V_1(V_1^2+V_2^2) \\ -3U_{p2}^2V_2 - 3U_{p2}V_2^2 - 2U_{p1}V_1(U_{p2}+V_2) \\ -U_{p1}^2V_2 - U_{p2}V_1^2 - V_2(V_1^2+V_2^2) \end{pmatrix}, \end{aligned}$$

hence we can set

$$G(V,W) = \begin{pmatrix} -3U_{p1}^2W_1 - 3U_{p1}V_1W_1 - 2U_{p2}W_2(U_{p1}+V_1) \\ -U_{p1}V_2W_2 - U_{p2}^2W_1 - W_1(V_1^2+V_2^2) \\ -3U_{p2}^2W_2 - 3U_{p2}V_2W_2 - 2U_{p1}W_1(U_{p2}+V_2) \\ -U_{p1}^2W_2 - U_{p2}V_1W_1 - W_2(V_1^2+V_2^2) \end{pmatrix}$$

for $W = (W_1, W_2)$.

As a preliminary step, we deal with (41) and (42) formally linearized about $(V,W) = (0,0)$. This system reads as

$$\left.\begin{aligned} \partial_t V &= BV + R_\alpha(x)V + DN(U_{\rm per})V, \\ \partial_t W &= \Lambda_\alpha W + DN(U_{\rm per})W; \end{aligned}\right\} \qquad (43)$$

note that since every term in $G(V,W)$ has a factor W it is found that $D_VG(0,0) = 0$, and we moreover have $D_WG(0,0) = D_V(N(U_{\rm per}+V) - N(U_{\rm per}))|_{V=0} = DN(U_{\rm per})$. For instance, in the above example we obtain

$$DN(U_{\rm per}) = \begin{pmatrix} -(3U_{p1}^2 + U_{p2}^2) & -2U_{p1}U_{p2} \\ -2U_{p1}U_{p2} & -(3U_{p2}^2 + U_{p1}^2) \end{pmatrix}.$$

According to (43) we need to study first the time-periodic operator

$$M_\alpha(t)W = \Lambda_\alpha W + DN(U_{\rm per}(t))W.$$

Lemma 13. *The operator M_α generates a linear evolution system $\mathcal{S}_{t,s}$ in $L^1(\mathbb{R}^d) \cap C_b^0(\mathbb{R}^d)$. There exist $\mu > 0$ and $C > 0$ such that*

$$\|\mathcal{S}_{t,s}\|_{L^1\cap C_b^0 \to L^1\cap C_b^0} \le Ce^{-\mu(t-s)}$$

for $s \le t$, and we have $\mu = \mathcal{O}(\varepsilon^2)$ as $\varepsilon \to 0$.

Proof: The second part $DN(U_{\rm per}(t))$ is of order $\mathcal{O}(\varepsilon)$, i.e., a small perturbation of Λ_α. Obviously, the part of the spectrum which is $\mathcal{O}(1)$-bounded away from the imaginary axis stays $\mathcal{O}(1)$ bounded away if the perturbation of order $\mathcal{O}(\varepsilon)$ is added. The two eigenvalues $\lambda_0^\pm(\alpha)$ with positive real parts are shifted into the left half plane, as in the case of a classical Hopf bifurcation, when linearizing the radial component of the reduced system

$$\partial_t r = \kappa_1 r + \kappa_3 r^3 + \mathcal{O}(\varepsilon^2)$$

about $r = \sqrt{\kappa_1/(-\kappa_3)}$. The deviation $\tilde{r} = r - \sqrt{\kappa_1/(-\kappa_3)}$ then satisfies

$$\partial_t \tilde{r} = -2\kappa_1 \tilde{r} + \mathcal{O}(\tilde{r}^2 + \varepsilon^2),$$

where we have used the assumption $\kappa_3 < 0$. Since $\mathcal{S}_{t,s}$ is generated by M_α which is the sum of a sectorial operator and a relatively compact perturbation in $L^1 \cap C_b^0$ with Floquet exponents strictly in the left half plane we have established the claim. □

Thus the second equation of (43) will lead to an exponentially (in time) damped solution W. Now we would like to proceed as in Section 2 and Section 3 and replace in the first equation of (43) all terms in V which have space-dependent coefficients by W. However, this is not possible in full analogy, since $U_{\text{per}}(x,t) = U_0(x) + \tilde{U}_{\text{per}}(x,t)$ with

$$\tilde{U}_{\text{per}}(x,t) = \sum_{n\in\mathbb{Z}\setminus\{0\}} U_n(x) e^{in\omega t},$$

but U_0 does not decay at an exponential rate. Accordingly we rewrite the first equation of (43) as

$$\begin{aligned}\partial_t V &= BV + R_\alpha(x)V + DN(U_0 + \tilde{U}_{\text{per}}(t))V \\ &= BV + DN(U_0)V + e^{-\beta x_1} R_\alpha(x) W \\ &\quad + e^{-\beta x_1}(DN(U_0 + \tilde{U}_{\text{per}}(t)) - DN(U_0))W.\end{aligned} \tag{44}$$

Lemma 14. *There exist $C > 0$ such that*

$$\|e^{-\beta x_1} R_\alpha\|_{C_b^0} + \|e^{-\beta x_1}(DN(U_0 + \tilde{U}_{\text{per}}(t)) - DN(U_0))\|_{C_b^0} \leq C.$$

Proof: The first bound is obtained from (H1), since we still have $\beta \leq c/2$ by the assumption on β in this section. In addition, identifying U and JU, it follows that

$$\begin{aligned}|DN(U_0 + \tilde{U}_{\text{per}}(t)) - DN(U_0)| &\leq C(|U_0| + |\tilde{U}_{\text{per}}(t)|)^{p-2}\,|\tilde{U}_{\text{per}}(t)| \\ &\leq C|\tilde{U}_{\text{per}}(t)|,\end{aligned}$$

cf. Theorem 2. But also

$$\begin{aligned}e^{-\beta x_1}|\tilde{U}_{\text{per}}(x,t)| &\leq e^{\beta|x|}\, e^{-\gamma|x|} \sum_{n\in\mathbb{Z}\setminus\{0\}} \|U_n e^{\gamma|x|}\|_{C_b^0} \\ &\leq C \sum_{n\in\mathbb{Z}\setminus\{0\}} \|U_n e^{\gamma|x|}\|_{H^m} \\ &= C \sum_{n\in\mathbb{Z}\setminus\{0\}} \|U_n\|_{H_\gamma^m} \leq C\|U\|_{\mathcal{Z}} \leq C,\end{aligned}$$

as a consequence of $\beta \leq \gamma$. □

Therefore, by Lemmas 13 and 14, the linearized equation (44) for V is of the form

$$\partial_t V = BV + DN(U_0)V + \mathcal{O}(e^{-\mu t}),$$

similar to the linearized system in Section 2, with the additional term $DN(U_0)V$ which will be handled similar to the nonlinear terms in the following.

After this preliminary investigation of the properties of the linearized system (43) we consider the full system. We rewrite (41) and (42) as

$$\left.\begin{aligned} \partial_t V &= BV + \Phi(x)V + e^{-\beta x_1} R_0(x,t)W + N_2(V), \\ \partial_t W &= M_\alpha(t)W + N_3(V,W), \end{aligned}\right\} \tag{45}$$

with

$$\begin{aligned} \Phi(x) &= DN(U_0(x)), \\ R_0(x,t) &= R_\alpha(x) + DN(U_0(x) + \tilde{U}_{\mathrm{per}}(x,t)) - DN(U_0(x)), \\ N_2(V) &= N(U_{\mathrm{per}} + V) - N(U_{\mathrm{per}}) - DN(U_{\mathrm{per}})V, \\ N_3(V,W) &= G(V,W) - DN(U_{\mathrm{per}})W \\ &= G(V,W) - G(0,0) - D_{V,W}G(0,0)(V,W), \end{aligned}$$

recalling $U_{\mathrm{per}} = U_0 + \tilde{U}_{\mathrm{per}}$. We apply the variation of constant formula to (45) and obtain

$$\left.\begin{aligned} V(t) &= e^{Bt}V(0) + \int_0^t e^{B(t-s)} \Big(\Phi(x)V(s) + e^{-\beta x_1} R_0(x,s)W(s) \\ &\qquad\qquad + N_2(V)(s)\Big)\, ds, \\ W(t) &= \mathcal{S}_{t,0}W(0) + \int_0^t \mathcal{S}_{t,s} N_3(V,W)(s)\, ds. \end{aligned}\right\} \tag{46}$$

In a way similar to Lemma 14 we obtain

Lemma 15. *There exists $C > 0$ such that the following estimates hold.*

$$\begin{aligned} \|N_2(V)\|_{L^1} &\le C\|V\|_{L^1}\|V\|_{L^\infty}, \\ \|N_2(V)\|_{L^\infty} &\le C\|V\|_{L^\infty}^2, \\ \|N_3(V,W)\|_{L^1\cap L^\infty} &\le C\|V\|_{L^\infty}\|W\|_{L^1\cap L^\infty}. \end{aligned}$$

Therefore the system (45) possesses exactly the same properties as the system (8), as soon as we can control the term

$$s_1(x,t) = \int_0^t e^{B(t-s)} \Big(\Phi(\cdot)V(s)\Big)(x)\, ds \tag{47}$$

in the L^1- and L^∞-norm similar to the nonlinear terms. Then we can proceed line by line as in Section 2 to complete the proof of Theorem 3. Hence it remains to give the

Estimates on s_1: We first note that $|\Phi(x)| \le C|U_0(x)|^{p-1}$, with p from (H2). Thus for a fixed $q < d/2$ the estimate

$$\|\Phi\|_{L^q} + \|\Phi\|_{L^\infty} \le C\|U_0\|_{L^\infty}^{p-2}\|U_0\|_{L^q} + C\|U_0\|_{L^\infty}^{p-1} \le \delta_0 \tag{48}$$

is obtained, where $\delta_0 > 0$ can be made as small as necessary, due to Remark 2. Next we define the functions $a(t)$ and $b(t)$ as in the proof of Theorem 1, and we modify $c(t)$ to

$$c(t) = \sup_{s\in[0,t]} \Big(e^{\sigma s} \|w(s)\|_{L^1\cap L^\infty} \Big),$$

with $\sigma = \mu/2$ and $\mu > 0$ as in Lemma 13. From (12) and Hölder's inequality we deduce that

$$\begin{aligned}
\|s_1\|_{L^1} &\le \int_0^t \|e^{B(t-s)}\|_{L^1\to L^1} \|\Phi(\cdot)V(s)\|_{L^1}\, ds \\
&\le C\|\Phi\|_{L^q} \int_0^t \|V(s)\|_{L^\infty}^{1/q} \|V(s)\|_{L^1}^{1/q'}\, ds \\
&\le C\|\Phi\|_{L^q}\, a(t)^{1/q} b(t)^{1/q'} \int_0^t (1+s)^{-d/(2q)}\, ds \\
&\le C\|\Phi\|_{L^q}\, a(t)^{1/q} b(t)^{1/q'},
\end{aligned}$$

due to $q < d/2$, where $q' = q/(q-1)$. In a similar manner we get

$$\begin{aligned}
&(1+t)^{d/2}\|s_1\|_{L^\infty} \\
&\le (1+t)^{d/2} \int_0^t \min\Big\{ \|e^{B(t-s)}\|_{L^1\to L^\infty} \|\Phi(\cdot)V(s)\|_{L^1}, \\
&\qquad\qquad \|e^{B(t-s)}\|_{L^\infty\to L^\infty} \|\Phi(\cdot)V(s)\|_{L^\infty} \Big\}\, ds \\
&\le C\|\Phi\|_{L^q} (1+t)^{d/2} \int_0^{t-1} (t-s)^{-d/2} \|V(s)\|_{L^\infty}^{1/q} \|V(s)\|_{L^1}^{1/q'}\, ds \\
&\quad + C\|\Phi\|_{L^\infty} (1+t)^{d/2} \int_{t-1}^t \|V(s)\|_{L^\infty}\, ds \\
&\le C\|\Phi\|_{L^\infty} a(t) \\
&\quad + C\|\Phi\|_{L^q} a(t)^{1/q} b(t)^{1/q'} (1+t)^{d/2} \\
&\qquad \times \int_0^{t-1} (t-s)^{-d/2} (1+s)^{-d/(2q)}\, ds \\
&\le C\|\Phi\|_{L^\infty} a(t) + C\|\Phi\|_{L^q} a(t)^{1/q} b(t)^{1/q'},
\end{aligned}$$

again due to $q < d/2$.

These estimates on s_1, the estimates on the linear semigroup in (12) and from Lemma 13, and the estimates on the nonlinear terms from Lemma 15 then give, as in Section 2 and completely analogous to (17),

$$\left.\begin{aligned}
a(t) &\le C_2\Big(a(0) + b(0) + \|\Phi\|_{L^\infty} a(t) + \|\Phi\|_{L^q} a(t)^{1/q} b(t)^{1/q'} \\
&\qquad a(t)^p + a(t)^{p-1} b(t) + c(t) \Big), \\
b(t) &\le C_2\Big(b(0) + \|\Phi\|_{L^q}\, a(t)^{1/q} b(t)^{1/q'} + a(t)^{p-1} b(t) + c(t) \Big), \\
c(t) &\le C_2\Big(c(0) + a(t)^{p-1} c(t) \Big),
\end{aligned}\right\} \tag{49}$$

with a constant $C_2 > 0$. Analogously to the proof of Theorem 1 this implies that given $\delta_1 > 0$ we may choose a sufficiently small $\delta_2 > 0$ (i.e., the magnitude of the perturbation), and then additionally a sufficiently small $\delta_0 > 0$ (i.e., the magnitude of the bifurcation parameter which controls the magnitudes of $\|\Phi\|_{L^q}$ and $\|\Phi\|_{L^\infty}$ by Remark 2) such that $a(0)+b(0)+c(0) \leq \delta_2$ leads to $a(t) + b(t) + c(t) \leq \delta_1$ for all $t \in [0, \infty[$. This finishes the proof of Theorem 3 in the case of (H1) and (H2). It is not hard to transfer the above stability proof to the case (H1)' and (H2)' for all $d \geq 2$. □

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