On a variational problem with lack of compactness related to the Strichartz inequality

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Abstract

We study a variational problem from nonlinear fiber optics which strongly lacks compactness, due to the absence of a priori bounds in spaces different from $L^2(\mathbb{R})$. A method is established how to restore this missing compactness by means of the dispersive properties inherent in the problem.

1 Introduction and main result

In this paper we consider the particular variational problem to minimize the functional

$$\varphi(u) = -\int_0^1 \int_{\mathbb{R}} |(e^{it\partial_x^2}u)(x)|^4 \, dx \, dt \tag{1.1}$$

under the constraint $u \in L^2 = L^2(\mathbb{R};\mathbb{C})$ and $\int_{\mathbb{R}} |u|^2 \, dx = \lambda$; here and henceforth $e^{it\partial_x^2}u_0 = U(t)u_0$ denotes the evolution operator of the free Schrödinger equation in one space dimension, i.e., $u(t,x) = (U(t)u_0)(x)$ solves

$$iu_t + u_{xx} = 0, \quad u(0,x) = u_0(x). \tag{1.2}$$

We will device a method to prove that a minimizer exists which also could be useful to handle similar problems associated to different dispersive equations. The functional φ arises in nonlinear optics, and critical points correspond to ground states of so-called (time-averaged) dispersion managed optical fibers, in the case of zero residual dispersion; cf. [17] and the references therein for a derivation and additional motivation. In general, minimizers of the functional

$$\varphi^{(\varepsilon)}(u) = \frac{\varepsilon}{2}\int_{\mathbb{R}} |u'|^2 \, dx - \int_0^1 \int_{\mathbb{R}} |(e^{it\partial_x^2}u)(x)|^4 \, dx \, dt \tag{1.3}$$

under the constraint $u \in H^1 = H^1(\mathbb{R};\mathbb{C})$ and $\int_{\mathbb{R}} |u|^2 \, dx = \lambda$ are interpreted as ground states at residual dispersion $\sim \varepsilon$, hence the functional $\varphi = \varphi^{(\varepsilon=0)}$ from (1.1) formally arises as the singular perturbation limit $\varepsilon \to 0$ of (1.3). It has been shown in [17, 9] that the constrained minimization problem associated with (1.3) admits a solution, cf. also [16].

After these general remarks, we return to (1.1) to see what is the main difficulty. If we take a minimizing sequence $(u_j) \subset L^2$, i.e., $\|u_j\|_{L^2}^2 = \lambda$ for $j \in \mathbb{N}$ and $\varphi(u_j) \to P_\lambda$ as $j \to \infty$, where

$$P_\lambda = \inf \Big\{ \varphi(u) : u \in L^2, \int_{\mathbb{R}} |u|^2 \, dx = \lambda \Big\}, \tag{1.4}$$

we may assume $u_j \rightharpoonup u$ in L^2, but there is no a priori reason why this sequence should converge strongly in L^2. We will prove, however, that the dispersive properties of (1.2) are good enough to establish this strong convergence.

Usually in variational problems defined on the whole real axis (or on the whole space) one tries to apply the concentration compactness principle (henceforth abbreviated CCP) for the sequence (u_j), cf. Section 3 below, to restore at least partial compactness, but without a H^1-bound on (u_j) this seems to lead nowhere. We note that in case of the functional $\varphi^{(\varepsilon)}$ from (1.3) such a H^1-bound is available, and resolving several technical issues, in [17] the CCP could finally be made use of successfully.

To get a clue on how to handle the problem for $\varepsilon = 0$, we first discuss the question of which additional degree of regularity for the minimizing sequence (besides L^2) would be sufficient to solve it. Since we are dealing with a L^2-problem, one could expect that a decomposition $u_j = u_j^{\text{reg}} + u_j^{\text{small}}$ into a H^1-part u_j^{reg} and a remainder u_j^{small} which is small in L^2 would allow to follow the H^1-proof for (1.3) 'up to ε'. However, at this point it remains unclear whether a relation between the size of $\|u_j^{\text{reg}}\|_{H^1}$ and the size of $\|u_j^{\text{small}}\|_{L^2}$ were needed. By refining some of the arguments in [17], this issue could be settled, and as a first step it was possible to show that basically no relation between $\|u_j^{\text{reg}}\|_{H^1}$ and $\|u_j^{\text{small}}\|_{L^2}$ is necessary to make the argument work. So we may reformulate our initial question as follows: For a L^2-bounded sequence (u_j), given $\varepsilon > 0$, what is the minimal additional regularity that allows a decomposition $u_j = u_j^{\text{reg}} + u_j^{\text{small}}$, with $\|u_j^{\text{small}}\|_{L^2} \sim \varepsilon$ and $\|u_j^{\text{reg}}\|_{H^1} \sim R_\varepsilon$, uniformly in $j \in \mathbb{N}$? To answer this question, we fix a function $\phi \in C_0^\infty(\mathbb{R})$ with values in $[0, 1]$ such that $\phi(\xi) = 1$ for $|\xi| \le 1$ and $\phi(\xi) = 0$ for $|\xi| \ge 2$. For $R > 0$ one defines $\phi_R(\xi) = \phi(\xi/R)$ to obtain

$$u_j = u_{j,R}^{\text{reg}} + u_{j,R}^{\text{small}}, \tag{1.5}$$

where

$$u_{j,R}^{\text{reg}} = (\phi_R \hat{u}_j)\check{} = \check{\phi}_R * u_j \quad \text{and} \quad u_{j,R}^{\text{small}} = ((1-\phi_R)\hat{u}_j)\check{} = (1-\phi_R)\check{} * u_j$$

for $j \in \mathbb{N}$ and $R > 0$. Then $\|u_{j,R}^{\text{reg}}\|_{\dot{H}^1} \sim R$ will be large, and we are going to argue on what is needed to get $\|u_{j,R}^{\text{small}}\|_{L^2}$ small. For this, we observe that

$$\|u_{j,R}^{\text{small}}\|_{L^2}^2 = \int_{|\xi| \ge R} |(1-\phi_R)\hat{u}_j|^2 \, d\xi \le \int_{|\xi| \ge R} |\hat{u}_j|^2 \, d\xi = 1 - \int_{|\xi| < R} |\hat{u}_j|^2 \, d\xi.$$

Hence the basic regularity assumption we are looking for is

$$\text{for every } \varepsilon > 0 \text{ there exists } R = R_\varepsilon > 0 : \quad \int_{|\xi| < R} |\hat{u}_j|^2 \, d\xi \ge 1 - \varepsilon, \quad j \in \mathbb{N}; \tag{1.6}$$

note that this assertion in particular holds in the case that $(u_j) \subset H^s$ is bounded for some $s > 0$.

We remark that with respect to dispersive equations the usefulness of decomposing functions u into a regular low-frequency part u^{reg} and a small high-frequency part u^{small} as in (1.5) has been observed for the first time in [2]. In this paper the global existence of solutions for critical nonlinear Schrödinger equations in two space dimensions is investigated, for data lying only in some H^s with

$3/5 < s < 1$, instead of H^1. The basic idea was that the problem could then be considered as a small perturbation $\sim u^{\text{small}}$ of the same problem with regular data $u^{\text{reg}} \in H^1$ leading to H^1-solutions, for which conservation of energy could be used. The approach from [2] meanwhile found a large number of applications and improvements, cf. [1, 3, 5, 7, 8, 14] and many other papers. Note that we intend to rely on an analogous method: In the background, there is the H^1-problem (1.3), and we try to 'ε-shadow' this problem to find a minimizer of (1.1) by decomposing the u_j. However, the difference here is that we cannot assume any H^s-regularity for the u_j.

Having (1.6) in mind therefore the next step is to see how to find a minimizing sequence which satisfies this condition. First we note that, by Ekeland's variational principle, we may select the minimizing sequence in such a way that additionally $\lambda^{-1}P_\lambda u_j + Q(u_j) \to 0$ in L^2 holds, where $Q(u)$ is related to the gradient of φ by $\varphi'(u) = -4Q(u)$, and

$$Q(u) = \int_0^1 U(-t)\Big(|U(t)u|^2(U(t)u)\Big)\,dt, \quad u \in L^2. \tag{1.7}$$

We note in passing that, with a slight abuse of notation, $Q(u) = Q(u,u,u)$, where the trilinear $Q(\cdot,\cdot,\cdot)$ is given by

$$Q(u_1,u_2,u_3) = \int_0^1 U(-t)\Big((U(t)u_1)(\overline{U(t)u_2})(U(t)u_3)\Big)\,dt, \quad u_1,u_2,u_3 \in L^2. \tag{1.8}$$

Thus a possibility to ensure (1.6) would be to verify that Q is slightly regularizing, in the sense that an estimate $\|Q(u)\|_{H^s} \le C\|u\|_{L^2}^3$ holds for some (small) $s > 0$. However, it follows from the Galilean invariance of the linear Schrödinger equation that such a bound cannot be satisfied, cf. Remark 2.3 below.

Nevertheless Q is regularizing in a special way, to be explained next. Condition (1.6) just says that the sequence $(\hat{u}_j)$ is tight, in the sense of measures. This property can be realized as the first possibility of a CCP alternative, but applied to the sequence $(\hat{u}_j)$ instead of (u_j). Hence it is sufficient to exclude the other two possibilities from the alternative in order to ensure (1.6). If the sequence $(\hat{u}_j)$ would not be tight, it were either 'vanishing', in the sense that it tends to zero in L^2_ξ uniformly over every interval of fixed length, or it were 'splitting' into two parts with widely separated supports. The first of these cases somehow corresponds to the situation that frequency interactions over fixed finite distances are suppressed, e.g. in estimates involving the formula for $\hat{Q}(u_j) := \widehat{Q(u_j)}$. Moreover, splitting has a similar effect, since one has to take into account only the contributions of frequencies which differ a lot from each other, due to the support separation of the two parts. It then turned out that if either vanishing or splitting is additionally supposed for $(\hat{u}_j)$, then improved estimates on $\hat{Q}(u_j)$ are possible which are strong enough to exclude these two possibilities in the end. Hence we could verify that in fact every minimizing sequence satisfies (1.6), up to rotation and translation of the original sequence.

Thus the strategy for the proof can be summarized as follows:

First apply CCP to $(\hat{u}_j)$. Then: (a) If $(\hat{u}_j)$ is tight, then (1.6) holds and the decomposition (1.5) can be introduced. This leads to 'almost minimizing sequences' $(u^{\text{reg}}_{j,R_k})_{j\in\mathbb{N}}$ which are H^1-bounded for every $k \in \mathbb{N}$; here a sequence $\varepsilon_k \to 0$ is fixed and $R_k = R_{\varepsilon_k}$ is chosen according to (1.6). Since now H^1-bounds are available it is then not difficult to prove, by means of a further application of the CCP now to (u_j), that (u_j) has a strongly convergent subsequence. (b) If $(\hat{u}_j)$ is 'vanishing', then study $\hat{Q}(u_j)$ which corresponds to the linearization of the functional in question, and which is a multiple convolution integral. Since frequency interactions over finite distances do not contribute much by assumption, one can show that $\hat{Q}(u_j) \to 0$ strongly in L^2, whence $\varphi(u_j) \to 0$ in contradiction to the fact that (u_j) is a minimizing sequence. (c) If $(\hat{u}_j)$ is 'splitting', then $\hat{u}_j \sim \hat{v}_j + \hat{w}_j$, and the supports of $\hat{v}_j$ and $\hat{w}_j$ are far apart. This can be used to prove, similarly to (b), that the 'cross terms' like $Q(v_j, v_j, w_j)$ in the expansion of $Q(u_j) \sim Q(v_j+w_j, v_j+w_j, v_j+w_j)$ are small, whence $Q(u_j) \sim Q(v_j) + Q(w_j)$ which in turn leads to $\varphi(u_j) \sim \varphi(v_j) + \varphi(w_j)$. Since either v_j or w_j has mass strictly less than 1 (uniformly in j), one can then derive a contradiction to the definition of P_1 in a standard way.

Upon elaborating the technical details, we obtain the following main result of this paper.

Theorem 1.1 *For every $\lambda > 0$ the minimization problem (1.4) admits a solution $u \in L^2 \cap L^\infty$.*

It should be noted that as a consequence of $P_\lambda = \lambda^2 P_1$ (see Lemma 2.6 below) it will suffice to verify that P_1 has a solution.

Although the main motivation to study (1.4) came from nonlinear optics, let us also mention that Theorem 1.1 can be reformulated as saying that the best constant $C_* = -P_1 > 0$ for the Strichartz-type estimate

$$\int_0^1 \int_{\mathbb{R}} |(e^{it\partial_x^2} u)(x)|^4 \, dx \, dt \le C \|u\|_{L^2}^4, \quad u \in L^2, \tag{1.9}$$

is attained at some function $u_* \in L^2$. We remark that a different proof of the key property (1.6) can be given which does not rely on the application of the CCP to the sequence $(\hat{u}_j)$. Taking this route would lead to a considerable shortening of several technical arguments, mainly in Section 2. However, this particular simplification is only possible due to the fact that (1.9) is a 'subcritical' estimate. Since we found the method of a two-level application of the CCP to be new, and since in particular a further refinement of this approach recently could be used to show that also the best constant in the standard 'critical' $L_t^6(L_x^6)$-Strichartz estimate

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |(e^{it\partial_x^2} u)(x)|^6 \, dx \, dt \le C \|u\|_{L^2}^6, \quad u \in L^2,$$

is attained [12], we decided to follow the general strategy of proof as outlined above.

The result of the present paper furthermore could be extended to prove that in the singular perturbation limit $\varepsilon \to 0$ minimizers $u^{(\varepsilon)}$ of the functional $\varphi^{(\varepsilon)}$ from (1.3) under the constraint $\int_{\mathbb{R}} |u^{(\varepsilon)}|^2 \, dx = \lambda$ converge strongly in L^2 (up to selection of a subsequence, shift, and rotation) to a mimimizer u of P_λ; see [11].

The paper is organized as follows. In Section 2 we collect some preliminary estimates, mainly of technical nature. Then in Section 3 we give some details on the CCP alternative in L^2, which seems to be unusual compared to the frequently found H^1-case. In Section 4 we finally carry out the proof of Theorem 1.1.

Concerning notation, we denote $L^p = L^p(\mathbb{R}; \mathbb{C})$ and $H^s = H^s(\mathbb{R}; \mathbb{C})$, with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^s}$, respectively. The inner product on L^2 is $(u, v)_{L^2} = \int_{\mathbb{R}} u\bar{v} \, dx$, whereas the Fourier transform of $u \in L^2$ is $\hat{u}(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ix\xi} u(x) \, dx$ with inverse $\check{u}$. By C we denote unimportant positive numerical constants, in particular the factors $(2\pi)^{-1/2}$ of $\hat{u}$ and $\check{u}$ will always be absorbed into some C.

2 Some auxiliary results and estimates

Since φ is defined as a space-time integral, it can be expected that the (standard) Strichartz estimate for (1.2) will provide useful.

Lemma 2.1 *Assume $q \in [4, \infty]$ and $r \in [2, \infty]$ are such that $2/q = 1/2 - 1/r$. Then*

$$\|U(\cdot)u\|_{L_t^q(L_x^r)} := \Big(\int_{\mathbb{R}} \|U(t)u\|_{L^r}^q \, dt \Big)^{1/q} \le C\|u\|_{L^2}, \quad u \in L^2,$$

with C depending only on q.

Proof: Cf. [4, Thm. 3.2.5(i)]. $\Box$

Note in particular that the pair $(q, r) = (4, 4)$ does not satisfy $2/q = 1/2 - 1/r$, and therefore $\varphi(u)$ from (1.1) in general could be infinite, if in its definition $\int_0^1 (\dots)\, dt$ were replaced by $\int_{\mathbb{R}} (\dots)\, dt$.

The next lemma states some elementary properties of φ and its linearization.

Lemma 2.2 *We have $\varphi \in C^1(L^2)$ and $\varphi'(u) = -4Q(u)$, where $Q(u)$ is given in (1.7). Then $\varphi(u) = -(Q(u), u)_{L^2}$, and the estimate*

$$\begin{aligned} \|Q(u_1, u_2, u_3) - Q(v_1, v_2, v_3)\|_{L^2} \ \le\ & C\Big(\|u_2\|_{L^2}\|u_3\|_{L^2}\|u_1 - v_1\|_{L^2} + \|u_3\|_{L^2}\|v_1\|_{L^2}\|u_2 - v_2\|_{L^2} \\ & + \|v_1\|_{L^2}\|v_2\|_{L^2}\|u_3 - v_3\|_{L^2}\Big) \end{aligned} \tag{2.1}$$

holds for $u_1, u_2, u_3, v_1, v_2, v_3 \in L^2$, with $Q(u_1, u_2, u_3)$ defined in (1.8). In particular, we have

$$\|Q(u) - Q(v)\|_{L^2} \le C\Big(\|u\|_{L^2} + \|v\|_{L^2}\Big)^2 \|u - v\|_{L^2}, \quad u, v \in L^2, \tag{2.2}$$

and moreover

$$\|Q(u)\|_{L^2} \le C\|u\|_{L^2}^3, \quad u \in L^2, \tag{2.3}$$

and

$$|\varphi(u) - \varphi(v)| \le C\Big(\|u\|_{L^2} + \|v\|_{L^2}\Big)^3 \|u - v\|_{L^2}, \quad u, v \in L^2. \tag{2.4}$$

The Fourier transform of $Q(u) \in L^2$ is given by

$$\hat{Q}(u)(\xi) = \widehat{Q(u)}(\xi) = Ci \int_{\mathbb{R}} \int_{\mathbb{R}} d\xi_1 d\xi_2 \Big(\frac{1 - e^{i\alpha(\xi, \xi_1, \xi_2)}}{\alpha(\xi, \xi_1, \xi_2)} \Big) \hat{u}(\xi - \xi_1 - \xi_2)\hat{\bar{u}}(\xi_1)\hat{u}(\xi_2), \tag{2.5}$$

where

$$\alpha(\xi, \xi_1, \xi_2) = \xi^2 - (\xi - \xi_1 - \xi_2)^2 + \xi_1^2 - \xi_2^2 = 2(\xi_1 + \xi_2)(\xi - \xi_2). \tag{2.6}$$

Proof: First, $(Q(u), u)_{L^2} = \int_{\mathbb{R}} Q(u)\bar{u}\, dx = \int_0^1 \int_{\mathbb{R}} |U(t)u|^4 \, dxdt = -\varphi(u)$ follows from

$$\int_{\mathbb{R}} (U(-t)f)\bar{g}\, dx = \int_{\mathbb{R}} f(\overline{U(t)g})\, dx, \quad f, g \in L^2. \tag{2.7}$$

For (2.1) we note that $Q(u_1, u_2, u_3) - Q(v_1, v_2, v_3) = Q(u_1 - v_1, u_2, u_3) + Q(v_1, u_2 - v_2, u_3) + Q(v_1, v_2, u_3 - v_3)$, hence it suffices to show (2.1) for $v_1 = v_2 = v_3 = 0$. Due to $\|U(t)f\|_{L^2} = \|f\|_{L^2}$, if we denote $u_k(t) = U(t)u_k$ for $k = 1, 2, 3$ and use Hölder's inequality first in the dx-integral and then in the dt-integral, we obtain

$$\begin{aligned} \|Q(u_1, u_2, u_3)\|_{L^2} &\leq \int_0^1 \left\|u_1(t)\overline{u_2(t)}u_3(t)\right\|_{L^2} dt \leq \int_0^1 \|u_1(t)\|_{L^6}\|u_2(t)\|_{L^6}\|u_3(t)\|_{L^6}\, dt \\ &\leq \|U(\cdot)u_1\|_{L^6_t(L^6_x)}\|U(\cdot)u_2\|_{L^6_t(L^6_x)}\|U(\cdot)u_3\|_{L^6_t(L^6_x)} \leq C\|u_1\|_{L^2}\|u_2\|_{L^2}\|u_3\|_{L^2}, \end{aligned} \tag{2.8}$$

where in the last step we have used Strichartz' inequality (cf. Lemma 2.1) with $q = r = 6$. Setting $u_1 = u_2 = u_3 = u$ and $v_1 = v_2 = v_3 = v$, (2.2) follows from (2.1), and (2.3) is (2.2) with $v = 0$. Concerning (2.4), we write $|\varphi(u) - \varphi(v)| = |(Q(u) - Q(v), u)_{L^2} + (Q(v), u - v)_{L^2}|$ and apply (2.2) and (2.3). Next, in view of $(\widehat{U(t)u})(\xi) = e^{-it\xi^2}\hat{u}(\xi)$ and $(\widehat{\overline{U(t)u}})(\xi) = e^{it\xi^2}\hat{\overline{u}}(\xi)$, the derivation of (2.5) is straightforward. Finally, in order to show $\varphi'(u) = -4Q(u)$ one expands $\varphi(u + h) - \varphi(u)$ and relies on estimates of the type used in (2.8). □

Remark 2.3 We include a few comments why an estimate of the type $\|Q(u)\|_{H^s} \leq C\|u\|_{L^2}^3$ for some $s > 0$ will not hold. For this we fix a function $u \in L^2$ and define $u_N(x) = e^{iNx}u(x)$ for $N \in \mathbb{N}$. Then $\|u_N\|_{L^2} = \|u\|_{L^2}$ is constant in N and $(U(t)u_N)(x) = e^{iNx}e^{-iN^2t}(U(t)u)(x - 2Nt)$ is found as the solution of the linear Schrödinger equation with initial data u_N. This implies that the function $Q(u_N)$ will be of the form $Q(u_N)(x) = e^{iNx}\phi_N(x)$, with the ϕ_N suitably well-behaved if u is chosen regular enough. Therefore we will obtain $\|Q(u_N)\|_{H^s} \sim N^s$ tending to infinity as $N \to \infty$.

Lemma 2.4 *Let $\mathbb{R} = \Omega_1 \cup \Omega_2$. Then*

$$\begin{aligned} &\Big|\int_0^1 \int_{\mathbb{R}} (U(t)u_1)(\overline{U(t)u_2})(U(t)u_3)(\overline{U(t)u_4})\, dxdt\Big| \\ &\leq C\Big(\|U(\cdot)u_{i_1}\|_{L^2([0,1]\times\Omega_1)} \prod_{\substack{k=1\\k\neq i_1}}^4 \|u_k\|_{L^2} + \|U(\cdot)u_{i_2}\|_{L^2([0,1]\times\Omega_2)} \prod_{\substack{k=1\\k\neq i_2}}^4 \|u_k\|_{L^2}\Big) \end{aligned}$$

for $u_1, u_2, u_3, u_4 \in L^2$ and any $i_1, i_2 \in \{1, \ldots, 4\}$. In particular,

$$\Big|\int_0^1 \int_{\mathbb{R}} (U(t)u_1)(\overline{U(t)u_2})(U(t)u_3)(\overline{U(t)u_4})\, dxdt\Big| \leq C\|u_1\|_{L^2}\|u_2\|_{L^2}\|u_3\|_{L^2}\|u_4\|_{L^2} \tag{2.9}$$

for $u_1, u_2, u_3, u_4 \in L^2$.

Proof: We consider any $\Omega \subset \mathbb{R}$, and we are going to show that integrating over $x \in \Omega$ the bound $\sim C\|U(\cdot)u_1\|_{L^2([0,1]\times\Omega)}\|u_2\|_{L^2}\|u_3\|_{L^2}\|u_4\|_{L^2}$ can be derived; the argument applies in an analogous way, if it is a $U(\cdot)u_k$ different from $U(\cdot)u_1$ which should get the $L^2([0,1]\times\Omega)$-norm. Using Hölder's inequality in the dx-integral and then in the dt-integral, we have

$$\begin{aligned} &\Big|\int_0^1 \int_{\Omega} (U(t)u_1)(\overline{U(t)u_2})(U(t)u_3)(\overline{U(t)u_4})\, dxdt\Big| \\ &\leq \int_0^1 \|U(t)u_1\|_{L^2(\Omega)}\|U(t)u_2\|_{L^6}\|U(t)u_3\|_{L^6}\|U(t)u_4\|_{L^6}\, dt \\ &\leq \Big(\int_0^1 \|U(t)u_1\|_{L^2(\Omega)}^2\, dt\Big)^{1/2} \|U(\cdot)u_2\|_{L^6_t(L^6_x)}\|U(\cdot)u_3\|_{L^6_t(L^6_x)}\|U(\cdot)u_4\|_{L^6_t(L^6_x)} \\ &\leq C\|U(\cdot)u_1\|_{L^2([0,1]\times\Omega)}\|u_2\|_{L^2}\|u_3\|_{L^2}\|u_4\|_{L^2}, \end{aligned}$$

the latter by Strichartz' estimate (cf. Lemma 2.1) with $q = r = 6$. $\square$

The functional φ is invariant under translation and rotation of u, in the following sense.

Lemma 2.5 *Let $x_0 \in \mathbb{R}$, $\xi_0 \in \mathbb{R}$, and $u \in L^2$. Then $\varphi(u(\cdot + x_0)) = \varphi(u)$ and $\varphi(e^{i\xi_0 x}u) = \varphi(u)$.*

Proof: For the translation invariance, we note that

$$(U(t)u(\cdot + x_0))(x) = (U(t)u)(x + x_0), \tag{2.10}$$

since both sides have Fourier transform $e^{ix_0\xi}e^{-it\xi^2}\hat{u}(\xi)$. Hence $\varphi(u(\cdot + x_0)) = \varphi(u)$ follows from the definition of φ, cf. (1.1). Next, defining $v(x) = e^{i\xi_0 x}u(x)$, we obtain $\hat{v}(\xi) = \hat{u}(\xi - \xi_0)$ and $\hat{\bar{v}}(\xi) = \hat{\bar{u}}(\xi + \xi_0)$. Thus (2.5) yields, by means of the transformation $\eta_1 = \xi_1 + \xi_0$, $\eta_2 = \xi_2 - \xi_0$, the relation

$$\begin{aligned}
\hat{Q}(v)(\xi) &= Ci \int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1 d\xi_2 \Big(\frac{1 - e^{i\alpha(\xi,\xi_1,\xi_2)}}{\alpha(\xi,\xi_1,\xi_2)}\Big)\, \hat{u}(\xi - \xi_1 - \xi_2 - \xi_0)\hat{\bar{u}}(\xi_1 + \xi_0)\hat{u}(\xi_2 - \xi_0) \\
&= Ci \int_{\mathbb{R}}\int_{\mathbb{R}} d\eta_1 d\eta_2 \Big(\frac{1 - e^{i\alpha(\xi,\eta_1-\xi_0,\eta_2+\xi_0)}}{\alpha(\xi,\eta_1 - \xi_0,\eta_2 + \xi_0)}\Big)\, \hat{u}(\xi - \xi_0 - \eta_1 - \eta_2)\hat{\bar{u}}(\eta_1)\hat{u}(\eta_2) \\
&= \hat{Q}(u)(\xi - \xi_0),
\end{aligned}$$

the latter in view of $\alpha(\xi,\eta_1-\xi_0,\eta_2+\xi_0) = 2(\eta_1-\xi_0+\eta_2+\xi_0)(\xi-\eta_2-\xi_0) = \alpha(\xi-\xi_0,\eta_1,\eta_2)$, recall (2.6). Therefore we find from $\hat{\bar{v}}(\xi) = \hat{\bar{u}}(\xi - \xi_0)$ that $\varphi(v) = -\int_{\mathbb{R}} Q(v)(x)\bar{v}(x)\,dx = -\int_{\mathbb{R}} \hat{Q}(v)(\xi)\hat{\bar{v}}(\xi)\,d\xi = -\int_{\mathbb{R}} \hat{Q}(u)(\xi - \xi_0)\hat{\bar{u}}(\xi - \xi_0)\,d\xi = \varphi(u)$. $\square$

Concerning the scaling of P_λ in λ, the following result holds which also shows that P_λ is finite and negative.

Lemma 2.6 *For $\lambda > 0$ we have $P_\lambda = \lambda^2 P_1$, and there are constants $C_1, C_2 > 0$ such that $-C_1\lambda^2 \leq P_\lambda \leq -C_2\lambda^2$ is satisfied. In particular,*

$$\varphi(u) \geq P_{\|u\|_{L^2}^2} = \|u\|_{L^2}^4 P_1, \quad u \in L^2. \tag{2.11}$$

Proof: If $u \in L^2$ is such that $\|u\|_{L^2}^2 = \lambda$, then $v(x) = \lambda^{-1/2}u(x)$ has $\|v\|_{L^2}^2 = 1$, and $\varphi(u) = \lambda^2\varphi(v)$. This implies $P_\lambda = \lambda^2 P_1$. Next, from (2.3) we find $|\varphi(u)| = |(Q(u), u)_{L^2}| \leq C_1\|u\|_{L^2}^4$, whence $P_\lambda \geq -C_1\lambda^2$. Concerning the upper bound, we consider the particular data function $v(x) = Ae^{-x^2/2}$, with $A > 0$ to be selected below; cf. [17, Appendix C]. Then $(U(t)v)(x) = \frac{A}{\sqrt{1+2it}}\, e^{-x^2/(2+4it)}$, and it follows that $\|v\|_{L^2}^2 = A^2\sqrt{\pi}$ as well as

$$-\varphi(v) = \int_0^1 \int_{\mathbb{R}} |U(t)v|^4\,dx\,dt = \sqrt{\frac{\pi}{2}}A^4 \int_0^1 \frac{dt}{1 + 4t^2} = \frac{\sqrt{\pi}}{2^{3/2}}\arctan(2)A^4.$$

To achieve $\|v\|_{L^2}^2 = \lambda$ we set $A^2 = \pi^{-1/2}\lambda$ and obtain $P_\lambda \leq \varphi(v) = -\frac{1}{2^{3/2}\sqrt{\pi}}\arctan(2)\lambda^2$, thus we can choose explicitly $C_2 = 2^{-3/2}\pi^{-1/2}\arctan(2) \cong 0.22$. $\square$

In the remaining part of this section we will derive some additional technical lemmas.

Lemma 2.7 *Let $(u_j) \subset L^2$ be bounded and such that for any $A > 0$*

$$\lim_{j\to\infty} \sup_{x_0\in\mathbb{R}} \int_{x_0-A}^{x_0+A} |u_j|^2\, dx = 0 \tag{2.12}$$

holds, i.e., (u_j) is 'vanishing'. With $\phi \in \mathcal{S}(\mathbb{R})$ (Schwartz functions) we define $u_j^{(l)} = (\phi\hat{u}_j)\check{} = \check{\phi} * u_j$. Then $(u_j^{(l)})$ is also vanishing.

Proof: Let $A > 0$ and $x_0 \in \mathbb{R}$. Then

$$\begin{aligned} I(A, x_0) \ &:= \ \int_{x_0-A}^{x_0+A} |u_j^{(l)}(x)|^2\, dx = \int_{x_0-A}^{x_0+A} dx \left| \int_{\mathbb{R}} \check{\phi}(y) u_j(x-y)\, dy \right|^2 \\ &\leq \ \int_{-A}^{A} dx \Big(\int_{\mathbb{R}} |\check{\phi}(y)||u_j(x+x_0-y)|\, dy \Big)^2. \end{aligned} \tag{2.13}$$

Since also $\check{\phi} \in \mathcal{S}(\mathbb{R})$, we have $|\check{\phi}(y)| \leq C(1+|y|)^{-2}$ for $y \in \mathbb{R}$. For fixed $R > 0$ it hence follows from Hölder's inequality that

$$\int_{|y|>R} |\check{\phi}(y)||u_j(x+x_0-y)|\, dy \leq CR^{-1} \int_{|y|>R} \frac{1}{1+|y|}\, |u_j(x+x_0-y)|\, dy \leq CR^{-1}. \tag{2.14}$$

On the other hand, in addition

$$\begin{aligned} \int_{|y|\leq R} |\check{\phi}(y)||u_j(x+x_0-y)|\, dy \ &\leq \ \|\check{\phi}\|_{L^\infty} \int_{|y|\leq R} |u_j(x+x_0-y)|\, dy \\ &\leq \ CR^{1/2} \Big(\int_{x+x_0-R}^{x+x_0+R} |u_j(z)|^2\, dz \Big)^{1/2} \end{aligned} \tag{2.15}$$

holds. Using (2.14) and (2.15) in (2.13), we find for any $R > 0$ the estimate

$$I(A, x_0) \leq CAR^{-2} + CR \int_{-A}^{A} dx \int_{x+x_0-R}^{x+x_0+R} |u_j(z)|^2\, dz \leq CAR^{-2} + CAR \int_{x_0-(A+R)}^{x_0+(A+R)} |u_j(z)|^2\, dz,$$

with C depending only on $\sup_{y\in\mathbb{R}} |\check{\phi}(y)|(1+|y|)^2$, $\|\check{\phi}\|_{L^\infty}$, and $\sup_{j\in\mathbb{N}} \|u_j\|_{L^2}$ (thus it would suffice that these quantities were bounded). Therefore

$$\sup_{x_0\in\mathbb{R}} \int_{x_0-A}^{x_0+A} |u_j^{(l)}|^2\, dx \leq CAR^{-2} + CAR \sup_{x_0\in\mathbb{R}} \int_{x_0-(A+R)}^{x_0+(A+R)} |u_j|^2\, dx. \tag{2.16}$$

Given $\varepsilon > 0$ we fix $R > 0$ large enough such that $CAR^{-2} \leq \varepsilon/2$. Then we apply (2.12) with A replaced by $A+R$ to find $j_0 \in \mathbb{N}$ with the property that the second term on the right-hand side of (2.16) is $\leq \varepsilon/2$ for $j \geq j_0$. Thus we obtain $\sup_{x_0\in\mathbb{R}} \int_{x_0-A}^{x_0+A} |u_j^{(l)}|^2\, dx \leq \varepsilon$ for $j \geq j_0$, and this yields the claim. □

Lemma 2.8 *For $u \in H^1$, $t \in [0,1]$, and $A \geq 1$ we have*

$$\int_{\mathbb{R}} |U(t)u|^4\, dx \leq C\Big(\sup_{x_0\in\mathbb{R}} \int_{x_0-2A}^{x_0+2A} |u|^2\, dx + A^{-1}\|u\|_{L^2}\|u\|_{H^1} \Big) \|u\|_{H^1}^2. \tag{2.17}$$

Proof: First we recall that for every $t \in \mathbb{R}$ and $A > 0$ the estimate

$$\int_{-A}^{A} |U(t)u|^2\,dx \le \int_{-2A}^{2A} |u|^2\,dx + CA^{-1}|t|\|u\|_{L^2}\|u'\|_{L^2} \tag{2.18}$$

holds, cf. [3, Lemma 8.33]; a similar result is in [17, Thm. 7.1], and see [10, Lemma 2.5] for the version stated here. To include a short proof, we fix $t > 0$, $A > 0$, and a function $\beta \in C_0^\infty(\mathbb{R}^n)$ taking values in $[0,1]$ such that $\beta(x) = 1$ for $|x| \le A$ and $\beta(x) = 0$ for $|x| \ge 2A$. Then we can suppose that $\|\beta'\|_{L^\infty} \le CA^{-1}$. Writing $u(t,x) = (U(t)u)(x)$, (1.2) shows that with $I(t) = \int_{\mathbb{R}} |u(t,x)|^2\beta(x)\,dx$ we have $\dot I(t) = (-2)\mathrm{Im}\int_{\mathbb{R}} \bar u(t,x)\partial_x u(t,x)\beta'(x)\,dx$. Therefore we obtain

$$\begin{aligned}
\int_{-A}^{A} |u(t,x)|^2\,dx &\le I(t) = I(0) + \int_0^t \dot I(s)\,ds \\
&\le \int_{-2A}^{2A} |u(x)|^2\,dx + CA^{-1}\int_0^t \|u(s)\|_{L^2}\|\partial_x u(s)\|_{L^2}\,ds \\
&= \int_{-2A}^{2A} |u(x)|^2\,dx + CA^{-1}t\,\|u\|_{L^2}\|u'\|_{L^2}\,,
\end{aligned}$$

where it has been used that (1.2) preserves every H^s-norm. Next we recall the general inequality

$$\int_{\mathbb{R}} |u|^4\,dx \le C\Big(\sup_{x_0\in\mathbb{R}} \int_{x_0-1}^{x_0+1} |u|^2\,dx\Big)\|u\|_{H^1}^2 \tag{2.19}$$

from [4, Lemma 8.3.7]; a proof of this is as follows. By the Sobolev embedding theorem we have $\|u\|_{L^\infty(]-1,1[)} \le C\|u\|_{H^1(]-1,1[)}$, and therefore by translation invariance $\|u\|_{L^\infty(]x_0-1,x_0+1[)} \le C\|u\|_{H^1(]x_0-1,x_0+1[)}$ for all $x_0 \in \mathbb{R}$, with $C > 0$ independent of x_0. This yields

$$\begin{aligned}
\int_{\mathbb{R}} |u|^4\,dx &= \sum_{k\in 2\mathbb{Z}} \int_{k-1}^{k+1} |u|^4\,dx \le \sum_{k\in 2\mathbb{Z}} \Big(\|u\|_{L^\infty(]k-1,k+1[)}^2 \int_{k-1}^{k+1} |u|^2\,dx\Big) \\
&\le C\Big(\sup_{x_0\in\mathbb{R}} \int_{x_0-1}^{x_0+1} |u|^2\,dx\Big) \sum_{k\in 2\mathbb{Z}} \|u\|_{H^1(]k-1,k+1[)}^2 = C\Big(\sup_{x_0\in\mathbb{R}} \int_{x_0-1}^{x_0+1} |u|^2\,dx\Big)\|u\|_{H^1}^2.
\end{aligned}$$

From (2.19) and (2.18), it follows with (2.10) that for $t \in [0,1]$ and $A \ge 1$ we have

$$\begin{aligned}
\int_{\mathbb{R}} |U(t)u|^4\,dx &\le C\Big(\sup_{x_0\in\mathbb{R}} \int_{x_0-1}^{x_0+1} |(U(t)u)(x)|^2\,dx\Big)\|U(t)u\|_{H^1}^2 \\
&= C\Big(\sup_{x_0\in\mathbb{R}} \int_{-1}^{1} |(U(t)u)(x_0+z)|^2\,dz\Big)\|u\|_{H^1}^2 \\
&\le C\Big(\sup_{x_0\in\mathbb{R}} \int_{-A}^{A} |U(t)u(x_0+\cdot)|^2\,dz\Big)\|u\|_{H^1}^2 \\
&\le C\Big(\sup_{x_0\in\mathbb{R}} \Big[\int_{-2A}^{2A} |u(x_0+\cdot)|^2\,dx + A^{-1}\|u(x_0+\cdot)\|_{L^2}\|u(x_0+\cdot)'\|_{L^2}\Big]\Big)\|u\|_{H^1}^2 \\
&\le C\Big(\sup_{x_0\in\mathbb{R}} \int_{x_0-2A}^{x_0+2A} |u|^2\,dx + A^{-1}\|u\|_{L^2}\|u\|_{H^1}\Big)\|u\|_{H^1}^2,
\end{aligned}$$

as was to be shown. □

Lemma 2.9 *Assume $u, v, w, h \in L^2$ are such that $u = v + w + h$ and $\|h\|_{L^2} \le 1$. Then*

$$\begin{aligned} |\varphi(u) - \varphi(v) - \varphi(w)| \;&\le\; C\Big(1 + \|u\|_{L^2}^3 + \|v\|_{L^2}^3 + \|w\|_{L^2}^3\Big)\|h\|_{L^2} \\ &\quad + C\Big(|\Lambda_1(v,w)| + |\Lambda_2(v,w)| + |\Lambda_3(v,w)| + \Lambda_4(v,w)\Big), \end{aligned}$$

with the remainder terms

$$\Lambda_1(v,w) \;=\; \int_0^1\int_{\mathbb{R}} (U(t)v)^2(\overline{U(t)w})^2\,dxdt, \quad \Lambda_2(v,w) = \int_0^1\int_{\mathbb{R}} |U(t)v|^2(U(t)v)(\overline{U(t)w})\,dxdt, \tag{2.20}$$

$$\Lambda_3(v,w) \;=\; \int_0^1\int_{\mathbb{R}} |U(t)w|^2(U(t)v)(\overline{U(t)w})\,dxdt, \quad \Lambda_4(v,w) = \int_0^1\int_{\mathbb{R}} |U(t)v|^2|U(t)w|^2\,dxdt. \tag{2.21}$$

Proof: By expanding $|v + w + h|^4$ for $u, v, h \in \mathbb{C}$, one obtains a formula for $\varphi(u)$ which is then divided into a part where at least one of the factors is h and a remainder part; cf. [17, (2)] for a similar calculation. The remainder part leads to the terms $\Lambda_1, \ldots, \Lambda_4$. On the other hand, the part with the h's consists of summands each of which can be written in either of the forms

$$\int_0^1\int_{\mathbb{R}} (U(t)f_1)(\overline{U(t)f_2})(U(t)f_3)(\overline{U(t)h})\,dxdt \quad \text{or} \quad \int_0^1\int_{\mathbb{R}} (\overline{U(t)f_1})(U(t)f_2)(\overline{U(t)f_3})(U(t)h)\,dxdt$$

for some $f_1, f_2, f_3 \in \{u, v, w, h\}$. Hence the desired bound on this part with the h's follows from (2.9) and $\|h\|_{L^2} \le 1$. □

Lemma 2.10 *For every $u \in L^2$ and $A > \delta > 0$ the estimate*

$$\int_{\mathbb{R}} |\hat{Q}(u)|^2\,d\xi \le C\|u\|_{L^2}^6(\delta + A^{-1/2}\delta^{-1/2}) + C\|u\|_{L^2}^5\,A\,\hat{\Gamma}(A)^{1/2} \tag{2.22}$$

holds, where

$$\hat{\Gamma}(A) := \sup_{\xi_0\in\mathbb{R}}\int_{\xi_0-A}^{\xi_0+A} |\hat{u}|^2\,d\xi, \quad A > 0,$$

denotes the concentration function of $\hat{u}$.

Proof: Defining $\Phi = \overline{\hat{Q}(u)}$, we have $\|\Phi\|_{L^2} = \|Q(u)\|_{L^2} \le C\|u\|_{L^2}^3$ by (2.3), and also

$$\begin{aligned} \int_{\mathbb{R}} |\hat{Q}(u)|^2\,d\xi \;&=\; \int_{\mathbb{R}} \hat{Q}(u)(\xi)\Phi(\xi)\,d\xi \\ &=\; Ci\int_{\mathbb{R}} d\xi\,\Phi(\xi)\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1 d\xi_2\left(\frac{1 - e^{i\alpha(\xi,\xi_1,\xi_2)}}{\alpha(\xi,\xi_1,\xi_2)}\right)\hat{u}(\xi - \xi_1 - \xi_2)\hat{\bar{u}}(\xi_1)\hat{u}(\xi_2) \end{aligned}$$

due to (2.5), with $\alpha(\xi, \xi_1, \xi_2) = 2(\xi_1 + \xi_2)(\xi - \xi_2)$. Using $|\frac{1}{\alpha}(1 - e^{i\alpha})| \le C(1 + |\alpha|)^{-1}$ and setting $\xi_3 = \xi - \xi_1 - \xi_2$, we hence obtain

$$\int_{\mathbb{R}} |\hat{Q}(u)|^2\,d\xi \le C\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1 d\xi_2 d\xi_3\left(\frac{1}{1 + |(\xi_1 + \xi_2)(\xi_1 + \xi_3)|}\right)\hat{\bar{u}}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\Phi(\xi_1 + \xi_2 + \xi_3), \tag{2.23}$$

where here and henceforth we omit the absolute values on the functions for notational simplicity. We divide the domain of integration of the integral on the right-hand side as follows. **Case 1:** $|\xi_1+\xi_2|\le\delta$ or $|\xi_1+\xi_3|\le\delta$. Since the integral is symmetric in ξ_2 and ξ_3, it suffices to consider e.g. $|\xi_1+\xi_2|\le\delta$. Then from Young's inequality, cf. [6, Cor. 4.5.2], it follows that

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1 d\xi_2 d\xi_3\, \mathbf{1}_{\{|\xi_1+\xi_2|\le\delta\}}\,(\ldots)\\
&\le \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1 d\eta\, d\xi_3\, \mathbf{1}_{\{|\eta|\le\delta\}}\,\hat{\bar{u}}(\xi_1)\hat{u}(\eta-\xi_1)\hat{u}(\xi_3)\Phi(\eta+\xi_3)\\
&\le C\|\hat{\bar{u}}*\hat{u}\|_{L^\infty}\,\|\hat{u}*\Phi(-\cdot)\|_{L^\infty}\delta \le C\|\hat{\bar{u}}\|_{L^2}\|\hat{u}\|_{L^2}\|\hat{u}\|_{L^2}\|\Phi(-\cdot)\|_{L^2}\delta \le C\|u\|_{L^2}^6\delta. \qquad (2.24)
\end{aligned}$$

Case 2: $|\xi_1+\xi_2|>\delta$ and $|\xi_1+\xi_3|>\delta$. **Case 2a:** $|\xi_1+\xi_2|\le A$ and $|\xi_1+\xi_3|\le A$. Here we have

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1 d\xi_2 d\xi_3\, \mathbf{1}_{\{\delta<|\xi_1+\xi_2|\le A,\,\delta<|\xi_1+\xi_3|\le A\}}\,(\ldots)\\
&\le C\int_{\mathbb{R}} d\xi_1\hat{\bar{u}}(\xi_1)\int_{\mathbb{R}} d\xi_2\, \mathbf{1}_{\{|\xi_1+\xi_2|\le A\}}\hat{u}(\xi_2)\int_{-\xi_1-A}^{-\xi_1+A} d\xi_3\,\hat{u}(\xi_3)\Phi(\xi_1+\xi_2+\xi_3)\\
&\le C\|\Phi\|_{L^2}\int_{\mathbb{R}} d\xi_1\hat{\bar{u}}(\xi_1)\int_{\mathbb{R}} d\xi_2\, \mathbf{1}_{\{|\xi_1+\xi_2|\le A\}}\hat{u}(\xi_2)\Big(\int_{-\xi_1-A}^{-\xi_1+A}|\hat{u}(\xi_3)|^2\,d\xi_3\Big)^{1/2}\\
&\le C\|u\|_{L^2}^3\,\hat{\Gamma}(A)^{1/2}\int_{\mathbb{R}} d\xi_1\hat{\bar{u}}(\xi_1)(\hat{u}*\mathbf{1}_{[-A,A]})(-\xi_1)\\
&\le C\|u\|_{L^2}^4\,\hat{\Gamma}(A)^{1/2}\|\hat{u}*\mathbf{1}_{[-A,A]}\|_{L^2}\le C\|u\|_{L^2}^4\,\hat{\Gamma}(A)^{1/2}\,\|\hat{u}\|_{L^2}\|\mathbf{1}_{[-A,A]}\|_{L^1}\\
&\le C\|u\|_{L^2}^5\,A\,\hat{\Gamma}(A)^{1/2}, \qquad (2.25)
\end{aligned}$$

once again by Young's inequality. **Case 2b:** $|\xi_1+\xi_2|>A$ or $|\xi_1+\xi_3|>A$. Since this case is again symmetric in ξ_2 and ξ_3, we may suppose that $|\xi_1+\xi_2|>A$. **Case 2b(i):** $|\xi_1+\xi_2|\ge|\xi_1+\xi_3|$. Then $1+|(\xi_1+\xi_2)(\xi_1+\xi_3)|\ge|\xi_1+\xi_2||\xi_1+\xi_3|\ge|\xi_1+\xi_2|^{1/2}|\xi_1+\xi_3|^{3/2}\ge A^{1/2}|\xi_1+\xi_3|^{3/2}$. Introducing the notation $k(\xi)=\mathbf{1}_{\{|\xi|>\delta\}}\,|\xi|^{-3/2}$ and $k_*(\xi)=k(\xi)(\hat{u}(-\cdot)*\Phi)(\xi)$, this yields

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1 d\xi_2 d\xi_3\, \mathbf{1}_{\{|\xi_1+\xi_2|>A,\,|\xi_1+\xi_3|>\delta,\,|\xi_1+\xi_2|\ge|\xi_1+\xi_3|\}}\,(\ldots)\\
&\le CA^{-1/2}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1 d\xi_2 d\xi_3\, \mathbf{1}_{\{|\xi_1+\xi_3|>\delta\}}\,|\xi_1+\xi_3|^{-3/2}\,\hat{\bar{u}}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\Phi(\xi_1+\xi_2+\xi_3)\\
&= CA^{-1/2}\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1 d\xi_3\, k(\xi_1+\xi_3)(\hat{u}(-\cdot)*\Phi)(\xi_1+\xi_3)\hat{\bar{u}}(\xi_1)\hat{u}(\xi_3)\\
&= CA^{-1/2}\int_{\mathbb{R}} d\xi_1\hat{\bar{u}}(\xi_1)(k_**\hat{u}(-\cdot))(\xi_1)\le CA^{-1/2}\|u\|_{L^2}\|k_**\hat{u}(-\cdot)\|_{L^2}\le CA^{-1/2}\|u\|_{L^2}^2\|k_*\|_{L^1}\\
&\le CA^{-1/2}\|u\|_{L^2}^2\|k\|_{L^1}\|\hat{u}(-\cdot)*\Phi\|_{L^\infty}\le CA^{-1/2}\|u\|_{L^2}^3\|\Phi\|_{L^2}\|k\|_{L^1}\\
&\le CA^{-1/2}\|u\|_{L^2}^6\delta^{-1/2} \qquad (2.26)
\end{aligned}$$

by further applications of Young's inequality and due to $\|k\|_{L^1}=4\delta^{-1/2}$. **Case 2b(ii):** $|\xi_1+\xi_3|\ge|\xi_1+\xi_2|$. Here we have $|\xi_1+\xi_3|\ge|\xi_1+\xi_2|>A$, thus $1+|(\xi_1+\xi_2)(\xi_1+\xi_3)|\ge A^{1/2}|\xi_1+\xi_2|^{3/2}$, and due to $|\xi_1+\xi_2|>\delta$ the reasoning of the previous case leads to the same bound (2.26) by exchanging ξ_3 and ξ_2. Taking together (2.24), (2.25), and (2.26), we obtain (2.22). □

Lemma 2.11 *Suppose $v, w \in L^2$ are such that, for some $\xi_0 \in \mathbb{R}$, $\delta > 0$, and $\xi_1^*, \xi_2^* \in \mathbb{R}$ with $\xi_2^* - \xi_1^* \geq 6\delta^{-1}$, we have $\hat{v}(\xi) = 0$ for $|\xi - \xi_0| \geq \xi_1^* + 2\delta^{-1}$ and $\hat{w}(\xi) = 0$ for $|\xi - \xi_0| \leq \xi_2^* - 2\delta^{-1}$. Then*

$$\Big|\int_{\mathbb{R}} Q(v,w,v)\bar{w}\,dx\Big| \leq C\|v\|_{L^2}^2\|w\|_{L^2}^2\delta^{1/3}, \tag{2.27}$$

$$\Big|\int_{\mathbb{R}} Q(v,v,v)\bar{w}\,dx\Big| \leq C\|v\|_{L^2}^3\|w\|_{L^2}\delta^{1/3}, \tag{2.28}$$

$$\Big|\int_{\mathbb{R}} Q(v,w,w)\bar{w}\,dx\Big| \leq C\|v\|_{L^2}\|w\|_{L^2}^3\delta^{1/3}, \tag{2.29}$$

$$\Big|\int_{\mathbb{R}} Q(v,v,w)\bar{w}\,dx\Big| \leq C\|v\|_{L^2}^2\|w\|_{L^2}^2\delta^{1/3}, \tag{2.30}$$

with $C > 0$ independent of ξ_0, δ, ξ_1^*, and ξ_2^*; recall (1.8) for the definition of $Q(\cdot,\cdot,\cdot)$.

Proof: We note that in general, analogously to (2.5),

$$\begin{aligned}\int_{\mathbb{R}} Q(u_1,u_2,u_3)\bar{u}_4\,dx &= \int_{\mathbb{R}} \widehat{Q}(u_1,u_2,u_3)\bar{\hat{u}}_4\,d\xi \\ &= Ci\int_{\mathbb{R}} d\xi\,\bar{\hat{u}}_4(\xi)\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1 d\xi_2\Big(\frac{1-e^{i\alpha(\xi,\xi_1,\xi_2)}}{\alpha(\xi,\xi_1,\xi_2)}\Big)\hat{u}_1(\xi-\xi_1-\xi_2)\hat{\bar{u}}_2(\xi_1)\hat{u}_3(\xi_2),\end{aligned}$$

with $\alpha(\xi,\xi_1,\xi_2) = 2(\xi_1+\xi_2)(\xi-\xi_2)$. Thus, as in the preceding Lemma 2.10,

$$\begin{aligned}&\Big|\int_{\mathbb{R}} Q(u_1,u_2,u_3)\bar{u}_4\,dx\Big| \\ &\leq C\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1 d\xi_2 d\xi_3\Big(\frac{1}{1+|(\xi_1+\xi_2)(\xi_1+\xi_3)|}\Big)\hat{\bar{u}}_2(\xi_1)\hat{u}_3(\xi_2)\hat{u}_1(\xi_3)\bar{\hat{u}}_4(\xi_1+\xi_2+\xi_3).\end{aligned} \tag{2.31}$$

First we are going to verify (2.27). For $u_1 = u_3 = v$ and $u_2 = u_4 = w$, we will bound the right-hand side of (2.31) in a way similar to the proof of Lemma 2.10. We fix $\eta > 0$. **Case 1:** $|\xi_1+\xi_2| \leq \eta$ or $|\xi_1+\xi_3| \leq \eta$. Note that the integral to be estimated is again symmetric in ξ_2 and ξ_3, thus we may restrict to the case $|\xi_1+\xi_2| \leq \eta$. But here

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1 d\xi_2 d\xi_3\,\mathbf{1}_{\{|\xi_1+\xi_2|\leq\eta\}}(\ldots) \leq C\|v\|_{L^2}^2\|w\|_{L^2}^2\eta \tag{2.32}$$

follows exactly by the argument leading to (2.24). **Case 2:** $|\xi_1+\xi_2| > \eta$ and $|\xi_1+\xi_3| > \eta$. **Case 2(i):** $|\xi_1+\xi_2| \geq |\xi_1+\xi_3|$. By assumption on the supports of $\hat{v}$ and $\hat{w}$, in order that $\hat{\bar{w}}(\xi_1)\hat{v}(\xi_2)\hat{v}(\xi_3)\bar{\hat{w}}(\xi_1+\xi_2+\xi_3)$ is non-zero, we must have

$$|\xi_1+\xi_0| \geq \xi_2^* - 2\delta^{-1}, \quad |\xi_2-\xi_0| \leq \xi_1^* + 2\delta^{-1}, \quad |\xi_3-\xi_0| \leq \xi_1^* + 2\delta^{-1}, \tag{2.33}$$

$$\text{and} \quad |\xi_1+\xi_2+\xi_3-\xi_0| \geq \xi_2^* - 2\delta^{-1}, \tag{2.34}$$

note $\hat{\bar{w}}(\xi_1) = \bar{\hat{w}}(-\xi_1)$. Therefore

$$\begin{aligned}1+|(\xi_1+\xi_2)(\xi_1+\xi_3)| &\geq |(\xi_1+\xi_2+\xi_3-\xi_0)+(\xi_0-\xi_3)|^{1/2}|\xi_1+\xi_3|^{3/2} \\ &\geq ([\xi_2^*-2\delta^{-1}]-[\xi_1^*+2\delta^{-1}])^{1/2}|\xi_1+\xi_3|^{3/2} \geq (2\delta^{-1})^{1/2}|\xi_1+\xi_3|^{3/2}.\end{aligned}$$

This yields

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1 d\xi_2 d\xi_3\, \mathbf{1}_{\{\eta<|\xi_1+\xi_3|\le|\xi_1+\xi_2|\}}\,(\dots)\\
&\le\ C\delta^{1/2}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1 d\xi_2 d\xi_3\, \mathbf{1}_{\{|\xi_1+\xi_3|>\eta\}}\,|\xi_1+\xi_3|^{-3/2}\,\hat{\bar{w}}(\xi_1)\hat{v}(\xi_2)\hat{v}(\xi_3)\hat{\bar{w}}(\xi_1+\xi_2+\xi_3)\\
&\le\ C\delta^{1/2}\eta^{-1/2}\|v\|_{L^2}^2\|w\|_{L^2}^2, \qquad (2.35)
\end{aligned}$$

the latter estimate being obtained analogously to (2.26). **Case 2(ii):** $|\xi_1+\xi_3|\ge|\xi_1+\xi_2|$. Here (2.34) and (2.33) imply $1+|(\xi_1+\xi_2)(\xi_1+\xi_3)|\ge|\xi_1+\xi_2|^{3/2}|(\xi_1+\xi_2+\xi_3-\xi_0)-(\xi_2-\xi_0)|^{1/2}\ge(2\delta^{-1})^{1/2}|\xi_1+\xi_2|^{3/2}$, hence $|\xi_1+\xi_2|>\eta$ shows that the same reasoning as in the previous case once more leads to the bound (2.35). Summarizing (2.32) and (2.35), we have seen that for any $\eta>0$ the bound

$$\Big|\int_{\mathbb{R}} Q(v,w,v)\bar{w}\,dx\Big|\le C\|v\|_{L^2}^2\|w\|_{L^2}^2(\eta+\delta^{1/2}\eta^{-1/2})$$

holds. Choosing $\eta=\delta^{1/3}$, we obtain (2.27).

Concerning (2.28), we use (2.31) with $u_1=u_2=u_3=v$ and $u_4=w$. Then the resulting expression is once more symmetric in ξ_2 and ξ_3. We distinguish cases as before and obtain the bound $C\|v\|_{L^2}^3\|w\|_{L^2}\eta$ in Case 1. For Case 2(i) and Case 2(ii), we need to have

$$|\xi_1+\xi_0|\le\xi_1^*+2\delta^{-1},\quad |\xi_2-\xi_0|\le\xi_1^*+2\delta^{-1},\quad |\xi_3-\xi_0|\le\xi_1^*+2\delta^{-1},$$
$$\text{and}\quad |\xi_1+\xi_2+\xi_3-\xi_0|\ge\xi_2^*-2\delta^{-1}$$

for $\hat{\bar{v}}(\xi_1)\hat{v}(\xi_2)\hat{v}(\xi_3)\hat{\bar{w}}(\xi_1+\xi_2+\xi_3)$ not to vanish. Thus $|\xi_1+\xi_2|\ge|\xi_1+\xi_3|$ implies $1+|(\xi_1+\xi_2)(\xi_1+\xi_3)|\ge|(\xi_1+\xi_2+\xi_3-\xi_0)-(\xi_3-\xi_0)|^{1/2}|\xi_1+\xi_3|^{3/2}\ge(2\delta^{-1})^{1/2}|\xi_1+\xi_3|^{3/2}$, and under the assumption $|\xi_1+\xi_3|\ge|\xi_1+\xi_2|$ we find $1+|(\xi_1+\xi_2)(\xi_1+\xi_3)|\ge|\xi_1+\xi_2|^{3/2}|(\xi_1+\xi_2+\xi_3-\xi_0)-(\xi_2-\xi_0)|^{1/2}\ge(2\delta^{-1})^{1/2}|\xi_1+\xi_2|^{3/2}$. Hence we can proceed as for (2.27) to verify (2.28).

Next, to see (2.29) we apply (2.31) with $u_1=v$ and $u_2=u_3=u_4=w$, and we only check the relevant estimates from Case 2(i) and Case 2(ii); note that although here the integral from (2.31) is not symmetric in ξ_2 and ξ_3, the estimates from Case 1 go through without problems, leading to the bound $C\|v\|_{L^2}\|w\|_{L^2}^3\eta$. Now we have

$$|\xi_1+\xi_0|\ge\xi_2^*-2\delta^{-1},\quad |\xi_2-\xi_0|\ge\xi_2^*-2\delta^{-1},\quad |\xi_3-\xi_0|\le\xi_1^*+2\delta^{-1},$$
$$\text{and}\quad |\xi_1+\xi_2+\xi_3-\xi_0|\ge\xi_2^*-2\delta^{-1}$$

as necessary conditions for $\hat{\bar{w}}(\xi_1)\hat{w}(\xi_2)\hat{v}(\xi_3)\hat{\bar{w}}(\xi_1+\xi_2+\xi_3)$ to be non-zero. In case that $|\xi_1+\xi_2|\ge|\xi_1+\xi_3|$ we once more have $1+|(\xi_1+\xi_2)(\xi_1+\xi_3)|\ge|(\xi_1+\xi_2+\xi_3-\xi_0)-(\xi_3-\xi_0)|^{1/2}|\xi_1+\xi_3|^{3/2}\ge(2\delta^{-1})^{1/2}|\xi_1+\xi_3|^{3/2}$. On the other hand, if $|\xi_1+\xi_3|\ge|\xi_1+\xi_2|$, then we use $|\xi_1+\xi_2|=|(\xi_1+\xi_2+\xi_3-\xi_0)+(\xi_0-\xi_3)|\ge(\xi_2^*-2\delta^{-1})-(\xi_1^*+2\delta^{-1})\ge2\delta^{-1}$ to bound $1+|(\xi_1+\xi_2)(\xi_1+\xi_3)|\ge|\xi_1+\xi_2|^2\ge(2\delta^{-1})^{1/2}|\xi_1+\xi_2|^{3/2}$. Hence we can again argue as before to show (2.29).

Finally, to prove (2.30), we use (2.31) with $u_1=u_2=v$ and $u_3=u_4=w$. Case 1 again goes through without problems, and for Case 2(i) and Case 2(ii) we note that here

$$|\xi_1+\xi_0|\le\xi_1^*+2\delta^{-1},\quad |\xi_2-\xi_0|\ge\xi_2^*-2\delta^{-1},\quad |\xi_3-\xi_0|\le\xi_1^*+2\delta^{-1},$$
$$\text{and}\quad |\xi_1+\xi_2+\xi_3-\xi_0|\ge\xi_2^*-2\delta^{-1}$$

have to be satisfied in order that $\hat{\bar{v}}(\xi_1)\hat{w}(\xi_2)\hat{v}(\xi_3)\hat{\bar{w}}(\xi_1+\xi_2+\xi_3)$ does not vanish. Since in the proof of (2.29) (cf. the last step) only the estimates $|\xi_3-\xi_0|\le\xi_1^*+2\delta^{-1}$ and $|\xi_1+\xi_2+\xi_3-\xi_0|\ge\xi_2^*-2\delta^{-1}$

have been used, and as these estimates can also be used here, we can proceed in the same way to deduce (2.30). □

Lemma 2.12 *We have $Q(u) \in L^\infty$ for $u \in L^2$, and*

$$\|Q(u) - Q(v)\|_{L^\infty} \le C\Big(\|u\|_{L^2} + \|v\|_{L^2}\Big)^2 \|u - v\|_{L^2}, \quad u, v \in L^2.$$

In particular, $\|Q(u)\|_{L^\infty} \le C\|u\|_{L^2}^3$ for $u \in L^2$.

Proof: From (1.7) and $\|U(t)f\|_{L^\infty} \le Ct^{-1/2}\|f\|_{L^1}$, cf. [4, Prop. 3.2.1], we obtain with $u(t) = U(t)u$ and $v(t) = U(t)v$, using Hölder's inequality in the dx-integral, that

$$\begin{aligned}
\|Q(u) - Q(v)\|_{L^\infty} &\le C\int_0^1 t^{-1/2} \Big\| |u(t)|^2 u(t) - |v(t)|^2 v(t) \Big\|_{L^1} dt \\
&\le C\int_0^1 t^{-1/2} \Big(\||u(t)|^2[u(t) - v(t)]\|_{L^1} + \|u(t)v(t)[\bar{u}(t) - \bar{v}(t)]\|_{L^1} \\
&\qquad + \||v(t)|^2[u(t) - v(t)]\|_{L^1} \Big) dt \\
&\le C\int_0^1 t^{-1/2} \Big(\|u(t)\|_{L^3}^2 \|u(t) - v(t)\|_{L^3} + \|u(t)\|_{L^3}\|v(t)\|_{L^3}\|u(t) - v(t)\|_{L^3} \\
&\qquad + \|v(t)\|_{L^3}^2 \|u(t) - v(t)\|_{L^3} \Big) dt \\
&\le \Big(\int_0^1 t^{-2/3}\, dt \Big)^{3/4} \Big(\|U(\cdot)u\|_{L_t^{12}(L_x^3)}^2 + \|U(\cdot)u\|_{L_t^{12}(L_x^3)} \|U(\cdot)v\|_{L_t^{12}(L_x^3)} \\
&\qquad + \|U(\cdot)v\|_{L_t^{12}(L_x^3)}^2 \Big) \|U(\cdot)(u - v)\|_{L_t^{12}(L_x^3)} \\
&\le C\Big(\|u\|_{L^2} + \|v\|_{L^2}\Big)^2 \|u - v\|_{L^2},
\end{aligned}$$

where in the last two steps we have applied Hölder's inequality to the dt-integral and then Strichartz' estimate (cf. Lemma 2.1) with $q = 12$ and $r = 3$. □

3 Concentration compactness in L^2

The 'concentration compactness principle' asserts that basically there are three possibilities for a (L^2 or H^1-) bounded sequence of functions: either it is tight (in the sense of measures), or it is 'vanishing' (it tends to zero uniformly on every interval of fixed length), or it is 'splitting' (into two parts with supports widely separated). This principle has found its clear formulation and a large number of applications through P.-L. Lions [13]. Since we will need the very explicit form of alternative (3), we include some details, following [4, Lemma 8.3.8], [15, Sect. 4.3], or [17, Lemma 6.1].

Lemma 3.1 *Let $(f_j) \subset L^2$ be a sequence such that $\|f_j\|_{L^2} = 1$ for $j \in \mathbb{N}$. Then there is a subsequence (not relabelled) such that exactly one of the following three possibilities occurs.*

(1) There exists a sequence $(z_j) \subset \mathbb{R}$ such that for every $\varepsilon > 0$ there is $R = R_\varepsilon > 0$ with the property that

$$\int_{z_j-R}^{z_j+R} |f_j|^2\,dz \geq 1-\varepsilon, \quad j \in \mathbb{N}.$$

(2) For every $A > 0$ we have

$$\lim_{j\to\infty} \sup_{z_0\in\mathbb{R}} \int_{z_0-A}^{z_0+A} |f_j|^2\,dz = 0.$$

(3) There is $\gamma \in]0,1[$ with the following property. For every $\delta \in]0,\gamma[$ there exist $j_0 = j_0(\delta) \in \mathbb{N}$ and $z_1^*, z_2^* \in \mathbb{R}$ with $z_2^* - z_1^* \geq 6\delta^{-1}$ such that

$$\gamma - \delta < \sup_{z_0\in\mathbb{R}} \int_{z_0-z_2^*}^{z_0+z_2^*} |f_j|^2\,dz < \gamma+\delta, \quad j \geq j_0, \tag{3.1}$$

and for every $j \geq j_0$ we may select $z_j \in \mathbb{R}$ satisfying

$$\gamma - \delta < \int_{z_j-z_1^*}^{z_j+z_1^*} |f_j|^2\,dz < \gamma+\delta. \tag{3.2}$$

In particular, if we fix functions $\rho, \theta \in C_0^\infty(\mathbb{R})$ with values in $[0,1]$ which satisfy $\rho(z) = 1$ for $|z| \leq z_1^*$, $\rho(z) = 0$ for $|z| \geq z_1^* + 2\delta^{-1}$, $\theta(z) = 0$ for $|z| \leq z_2^* - 2\delta^{-1}$, and $\theta(z) = 1$ for $|z| \geq z_2^*$, then defining $v_j(z) = \rho(z-z_j)f_j(z)$ and $w_j(z) = \theta(z-z_j)f_j(z)$ one obtains for $j \geq j_0$ the estimates

$$\|f_j - (v_j+w_j)\|_{L^2}^2 \leq 2\delta, \quad \Big|\|v_j\|_{L^2}^2 - \gamma\Big| \leq 3\delta, \quad \text{and} \quad \Big|\|w_j\|_{L^2}^2 - (1-\gamma)\Big| \leq 9\delta.$$

Proof: The argument relies on the Lévy concentration functions $\Gamma_j(z) = \sup_{z_0\in\mathbb{R}} \int_{z_0-z}^{z_0+z} |f_j(y)|^2\,dy$. Then $0 \leq \Gamma_j(z) \leq 1$ and Γ_j is non-decreasing. Hence there exists a subsequence of (f_j), a countable set $E \subset \mathbb{R}$, and a non-negative and non-decreasing function Γ such that $\Gamma_j(z) \to \Gamma(z)$ as $j \to \infty$ for every $z \in \mathbb{R}\setminus E$. With $\gamma := \lim_{z\to\infty} \Gamma(z) \in [0,1]$, there are three possibilities: the cases $\gamma = 1$ or $\gamma = 0$ lead to alternative (1) or (2), respectively, cf. [4, Lemma 8.3.8] (here it is not needed that (f_j) is bounded in H^1). So it remains to show that $\gamma \in]0,1[$ implies (3). To see this, we fix $\delta \in]0,\gamma[$ and choose $z^* \in \mathbb{R}$ such that $\gamma - \delta < \Gamma(z) \leq \gamma$ for $z \geq z^*$. Then we take two widely separated points where Γ_j converges to Γ, i.e., we take $z_1^*, z_2^* \in (\mathbb{R}\setminus E) \cap [z^*, \infty[$ with $z_2^* - z_1^* \geq 6\delta^{-1}$ and $j_0 \in \mathbb{N}$ such that $\gamma - \delta < \Gamma_j(z_1^*) \leq \Gamma_j(z_2^*) < \gamma + \delta$ for $j \geq j_0$. By definition of Γ_j, this yields (3.1), and moreover for every $j \geq j_0$ we find $z_j \in \mathbb{R}$ such that (3.2) holds. With ρ and θ as in (3), we then define v_j and w_j. In view of (3.1) and (3.2) we have

$$\int_{z_1^*\leq|z-z_j|\leq z_2^*} |f_j(z)|^2\,dz = \int_{z_j-z_2^*}^{z_j+z_2^*} |f_j|^2\,dz - \int_{z_j-z_1^*}^{z_j+z_1^*} |f_j|^2\,dz \leq \gamma+\delta-(\gamma-\delta) = 2\delta. \tag{3.3}$$

Due to the support properties of v_j and w_j therefore

$$\|f_j - (v_j+w_j)\|_{L^2}^2 \leq \int_{z_1^*\leq|z-z_j|\leq z_2^*} |f_j(z)|^2\,dz \leq 2\delta.$$

In addition, (3.2), $z_1^* + 2\delta^{-1} \le z_2^*$, and (3.3) imply

$$\Big|\|v_j\|_{L^2}^2 - \gamma\Big| = \Big| \int_{z_j - z_1^*}^{z_j + z_1^*} |f_j(z)|^2 \, dz - \gamma + \int_{z_1^* \le |z - z_j| \le z_1^* + 2\delta^{-1}} \rho(z - z_j)^2 |f_j(z)|^2 \, dz \Big| \le 3\delta.$$

Finally, this and $\|f_j\|_{L^2} = 1$ in turn yield

$$\begin{aligned}
\Big|\|w_j\|_{L^2}^2 - (1-\gamma)\Big| &\le 3\delta + \Big|\|w_j\|_{L^2}^2 + \|v_j\|_{L^2}^2 - \|f_j\|_{L^2}^2\Big| \\
&= 3\delta + \Big| \int_{|z - z_j| > z_2^*} |f_j(z)|^2 \, dz + \int_{z_2^* - 2\delta^{-1} \le |z - z_j| \le z_2^*} \theta(z - z_j)^2 |f_j(z)|^2 \, dz \\
&\qquad + \int_{|z - z_j| < z_1^*} |f_j(z)|^2 \, dz + \int_{z_1^* \le |z - z_j| \le z_1^* + 2\delta^{-1}} \rho(z - z_j)^2 |f_j(z)|^2 \, dz \\
&\qquad - \int_{\mathbb{R}} |f_j(z)|^2 \, dz \Big| \\
&\le 3\delta + \int_{z_1^* \le |z - z_j| \le z_2^*} |f_j(z)|^2 \, dz + \int_{z_2^* - 2\delta^{-1} \le |z - z_j| \le z_2^*} |f_j(z)|^2 \, dz \\
&\qquad + \int_{z_1^* \le |z - z_j| \le z_1^* + 2\delta^{-1}} |f_j(z)|^2 \, dz \le 9\delta,
\end{aligned}$$

where we have once more used $z_1^* + 2\delta^{-1} \le z_2^*$ and (3.3). □

4 Proof of Theorem 1.1

We consider a minimizing sequence for P_1, i.e., $(u_j) \subset L^2$ such that $\|u_j\|_{L^2} = 1$ for $j \in \mathbb{N}$, and $\varphi(u_j) \to P_1$ as $j \to \infty$. Then also $\|\hat{u}_j\|_{L^2} = 1$ for $j \in \mathbb{N}$, whence we can apply Lemma 3.1 to the sequence $(f_j) = (\hat{u}_j)$. The following three subsections 4.1, 4.2, and 4.3 deal with the three possibilities (1), (2), and (3) which then may occur according to Lemma 3.1.

4.1 Case (1): The Fourier transforms are tight

We assume that there is a sequence $(\xi_j) \subset \mathbb{R}$ such that for each $\varepsilon > 0$ we may select $R = R_\varepsilon > 0$ satisfying $\int_{\xi_j - R}^{\xi_j + R} |\hat{u}_j|^2 \, d\xi \ge 1 - \varepsilon$ for $j \in \mathbb{N}$. Letting $v_j(x) = e^{-i\xi_j x} u_j(x)$, we obtain $\|v_j\|_{L^2} = 1$ and $\varphi(v_j) = \varphi(u_j)$ by Lemma 2.5. Hence (v_j) is a minimizing sequence as well. In addition, $\hat{v}_j(\xi) = \hat{u}_j(\xi + \xi_j)$ leads to

$$\int_{\xi_j - R}^{\xi_j + R} |\hat{u}_j(\xi)|^2 \, d\xi = \int_{-R}^{R} |\hat{u}_j(\xi + \xi_j)|^2 \, d\xi = \int_{-R}^{R} |\hat{v}_j(\xi)|^2 \, d\xi.$$

This argument shows that, passing to (v_j) if necessary, w.l.o.g. we may suppose $\xi_j = 0$ for all $j \in \mathbb{N}$, i.e.,

$$\forall\, \varepsilon > 0 \quad \exists\, R = R_\varepsilon > 0 : \quad \int_{|\xi| < R} |\hat{u}_j|^2 \, d\xi \ge 1 - \varepsilon, \quad j \in \mathbb{N}; \tag{4.1}$$

this is the basic assumption of Section 4.1, and it is used to improve the original sequence (u_j) as follows. We choose $\phi \in C_0^\infty(\mathbb{R})$ with values in $[0,1]$ such that $\phi(\xi) = 1$ for $|\xi| \le 1$ and $\phi(\xi) = 0$ for

$|\xi| \geq 2$, and for $R > 0$ we define $\phi_R(\xi) = \phi(\xi/R)$. With a fixed sequence $\varepsilon_k \to 0$ as $k \to \infty$, we select $R_k := R_{\varepsilon_k}$ corresponding to ε_k via (4.1) for $k \in \mathbb{N}$. Setting $\phi_k = \phi_{R_k}$, we decompose every u_j in a regular low frequency part and an L^2-small high frequency part as

$$u_j = u_{j,k}^{(l)} + u_{j,k}^{(h)}, \quad j,k \in \mathbb{N}, \tag{4.2}$$

where

$$u_{j,k}^{(l)} = (\phi_k \hat{u}_j)\check{} = \check{\phi}_k * u_j \quad \text{and} \quad u_{j,k}^{(h)} = ((1-\phi_k)\hat{u}_j)\check{} = (1-\phi_k)\check{} * u_j. \tag{4.3}$$

Then for all $j, k \in \mathbb{N}$,

$$\|u_{j,k}^{(l)}\|_{\dot{H}^1}^2 = \int_{\mathbb{R}} |\xi|^2 |\phi_k \hat{u}_j|^2 \, d\xi = \int_{|\xi| \leq 2R_k} |\xi|^2 |\phi_k \hat{u}_j|^2 \, d\xi \leq C R_k^2 \|\hat{u}_j\|_{L^2}^2 = C R_k^2 \,, \tag{4.4}$$

and also, by (4.1),

$$\|u_{j,k}^{(h)}\|_{L^2}^2 = \int_{|\xi| \geq R_k} |(1-\phi_k)\hat{u}_j|^2 \, d\xi \leq \int_{|\xi| \geq R_k} |\hat{u}_j|^2 \, d\xi = 1 - \int_{|\xi| < R_k} |\hat{u}_j|^2 \, d\xi \leq \varepsilon_k \,. \tag{4.5}$$

In addition,

$$\|u_{j,k}^{(l)}\|_{L^2} \leq \|\hat{u}_j\|_{L^2} = 1. \tag{4.6}$$

The next step consists in application of Lemma 3.1 to $(f_j) = (u_j)$, yielding a further subsequence $(u_{j'}) \subset (u_j)$, which however will be not relabelled for notational convenience. The argument is divided further according to which one of the possibilities (1), (2), or (3) from Lemma 3.1 arises.

4.1.1 The case that the sequence (u_j) is tight

We suppose that (1) in Lemma 3.1 occurs for (u_j), i.e., there is a sequence $(x_j) \subset \mathbb{R}$ such that for every $\delta > 0$ we find $M = M_\delta$ satisfying $\int_{x_j - M}^{x_j + M} |u_j|^2 \, dx \geq 1 - \delta$ for $j \in \mathbb{N}$. Setting $v_j(x) = u_j(x_j + x)$, we have $\|v_j\|_{L^2} = 1$ as well as $\varphi(v_j) = \varphi(u_j)$, by Lemma 2.5. Thus (v_j) is a minimizing sequence. Moreover, $\hat{v}_j(\xi) = e^{ix_j\xi}\hat{u}_j(\xi)$, whence (4.1) is satisfied for $\hat{v}_j$, too, and we could perform the same decomposition as above with v_j in place of u_j, leading once more to (4.4) and (4.5). Finally,

$$\int_{x_j - M}^{x_j + M} |u_j(x)|^2 \, dx = \int_{-M}^{M} |u_j(x + x_j)|^2 \, dx = \int_{-M}^{M} |v_j(x)|^2 \, dx.$$

Hence we could consider (v_j) instead of (u_j), thereby achieving $x_j = 0$ for all $j \in \mathbb{N}$. Thus we assume w.l.o.g. that

$$\forall\, \delta > 0 \quad \exists\, M = M_\delta > 0 : \quad \int_{|x| < M} |u_j|^2 \, dx \geq 1 - \delta, \quad j \in \mathbb{N}. \tag{4.7}$$

Since $(u_j) \subset L^2$ and $\|u_j\|_{L^2} = 1$, we can suppose that $u_j \rightharpoonup u$ in L^2 as $j \to \infty$ for some $u \in L^2$ with $\|u\|_{L^2} \leq 1$. We fix $k \in \mathbb{N}$ and observe that, due to (4.4) and (4.6), $(u_{j,k}^{(l)})_{j \in \mathbb{N}}$ is bounded in H^1 (by $\sim R_k$). Additionally, (4.5) implies that $(u_{j,k}^{(h)})_{j \in \mathbb{N}}$ is bounded in L^2. Thus we can select a subsequence $(j') \subset \mathbb{N}$, depending on k, and $v_k \in H^1$ as well as $w_k \in L^2$ satisfying $u_{j',k}^{(l)} \rightharpoonup v_k$ in H^1 and $u_{j',k}^{(h)} \rightharpoonup w_k$ in L^2 as $j' \to \infty$. In particular,

$$\|w_k\|_{L^2} \leq \liminf_{j' \to \infty} \|u_{j',k}^{(h)}\|_{L^2} \leq \varepsilon_k^{1/2} \tag{4.8}$$

by (4.5), and also $u = v_k + w_k$ in view of (4.2).

Next we fix $\delta > 0$ and choose $M = M_\delta$ according to (4.7). By compactness of the embedding $H^1 \subset L^2(]-M, M[)$, we see that $u^{(l)}_{j',k} \to v_k$ in $L^2(]-M, M[)$ as $j' \to \infty$. Hence we obtain from (4.8), (4.2), (4.5) and (4.7) that

$$\begin{aligned}
\|u\|_{L^2} &\geq \|u\|_{L^2(]-M,M[)} = \|v_k + w_k\|_{L^2(]-M,M[)} \geq \|v_k\|_{L^2(]-M,M[)} - \|w_k\|_{L^2} \\
&\geq \lim_{j'\to\infty} \|u^{(l)}_{j',k}\|_{L^2(]-M,M[)} - \varepsilon_k^{1/2} = \lim_{j'\to\infty} \|u_{j'} - u^{(h)}_{j',k}\|_{L^2(]-M,M[)} - \varepsilon_k^{1/2} \\
&\geq \limsup_{j'\to\infty} \Big(\|u_{j'}\|_{L^2(]-M,M[)} - \|u^{(h)}_{j',k}\|_{L^2}\Big) - \varepsilon_k^{1/2} \\
&\geq \limsup_{j'\to\infty} \|u_{j'}\|_{L^2(]-M,M[)} - 2\varepsilon_k^{1/2} \geq (1-\delta)^{1/2} - 2\varepsilon_k^{1/2}.
\end{aligned}$$

Taking successive limits $\delta \to 0$ and $k \to \infty$, it follows that $\|u\|_{L^2} = 1 = \lim_{j\to\infty} \|u_j\|_{L^2}$, and thus $u_j \rightharpoonup u$ in L^2 leads to $u_j \to u$ in L^2. Since $\varphi : L^2 \to \mathbb{R}$ is continuous, cf. (2.4), and (u_j) is a minimizing sequence, thus φ has a minimizer, i.e., P_1 admits a solution.

4.1.2 The case that the sequence (u_j) is vanishing

Throughout this subsection we assume that (1) in Lemma 3.1 holds for $(\hat{u}_j)$, whereas (2) in Lemma 3.1 occurs for (u_j), i.e.,

$$\lim_{j\to\infty} \sup_{x_0\in\mathbb{R}} \int_{x_0-A}^{x_0+A} |u_j|^2\,dx = 0 \tag{4.9}$$

is satisfied for every $A > 0$. Once again we will rely on decomposing the u_j in low and high frequencies, cf. (4.2). For $k \in \mathbb{N}$ fixed we claim that

$$\lim_{j\to\infty} \varphi(u^{(l)}_{j,k}) = 0 \tag{4.10}$$

holds. To verify this, we first note that (4.9) in conjunction with Lemma 2.7 implies that also

$$\lim_{j\to\infty} \sup_{x_0\in\mathbb{R}} \int_{x_0-A}^{x_0+A} |u^{(l)}_{j,k}|^2\,dx = 0 \tag{4.11}$$

is satisfied for all $A > 0$, since k is fixed and $\phi_k \in C_0^\infty(\mathbb{R})$. As $u^{(l)}_{j,k} \in H^1$, we may invoke Lemma 2.8, and integrating (2.17) with $u = u^{(l)}_{j,k}$ over $t \in [0,1]$ it follows that for $A \geq 1$

$$\begin{aligned}
|\varphi(u^{(l)}_{j,k})| &= \int_0^1 \int_{\mathbb{R}} |U(t)u^{(l)}_{j,k}|^4\,dxdt \leq C\Big(\sup_{x_0\in\mathbb{R}} \int_{x_0-2A}^{x_0+2A} |u^{(l)}_{j,k}|^2\,dx + A^{-1}\|u^{(l)}_{j,k}\|_{L^2}\|u^{(l)}_{j,k}\|_{H^1}\Big)\|u^{(l)}_{j,k}\|_{H^1}^2 \\
&\leq C\Big(\sup_{x_0\in\mathbb{R}} \int_{x_0-2A}^{x_0+2A} |u^{(l)}_{j,k}|^2\,dx + A^{-1}R_k\Big)R_k^2,
\end{aligned}$$

where we have used (4.6) and (4.4) in the last step. Taking into account (4.11), we thus have shown (4.10). Next, according to (2.4), (4.2), (4.6) and (4.5) it follows that

$$\begin{aligned}
|\varphi(u_j)| &\leq |\varphi(u_j) - \varphi(u^{(l)}_{j,k})| + |\varphi(u^{(l)}_{j,k})| \leq C\Big(\|u_j\|_{L^2} + \|u^{(l)}_{j,k}\|_{L^2}\Big)^3 \|u^{(h)}_{j,k}\|_{L^2} + |\varphi(u^{(l)}_{j,k})| \\
&\leq C\varepsilon_k^{1/2} + |\varphi(u^{(l)}_{j,k})|
\end{aligned}$$

for every $j, k \in \mathbb{N}$. Whence (4.10) implies $\limsup_{j\to\infty} |\varphi(u_j)| \leq C\varepsilon_k^{1/2}$ for all $k \in \mathbb{N}$, and therefore $P_1 = \lim_{j\to\infty} \varphi(u_j) = 0$, in contradiction to Lemma 2.6. Thus the case considered in this Section 4.1.2 in fact cannot occur.

4.1.3 The case that the sequence (u_j) is splitting

In this subsection we suppose that we have (1) in Lemma 3.1 for $(\hat{u}_j)$, but (3) in Lemma 3.1 holds for (u_j). Thus we can once more use the decomposition of the u_j in low and high frequencies, recall (4.2), but additionally we have $\gamma \in]0,1[$, where $\gamma = \lim_{x\to\infty} \Gamma(x)$. Here $\Gamma(x) = \lim_{j\to\infty} \Gamma_j(x) = \lim_{j\to\infty} \sup_{x_0\in\mathbb{R}} \int_{x_0-x}^{x_0+x} |u_j|^2\, dy$ is the pointwise (outside a countable set) limit of the concentration functions corresponding to (u_j). We now fix $\delta \in]0,\gamma[$ and choose $j_0 \in \mathbb{N}$, $x_1^* = z_1^* \in \mathbb{R}$, $x_2^* = z_2^* \in \mathbb{R}$, $x_j = z_j$ for $j \geq j_0$, and moreover the functions ρ and θ as described in (3) of Lemma 3.1; all these quantities are depending on δ. Defining $v_j(x) = \rho(x-x_j)u_j(x)$ and $w_j(x) = \theta(x-x_j)u_j(x)$, we recall that then $\|v_j\|_{L^2} \leq 1$, $\|w_j\|_{L^2} \leq 1$, and also

$$\|u_j - (v_j + w_j)\|_{L^2}^2 \leq 2\delta, \quad \Big|\|v_j\|_{L^2}^2 - \gamma\Big| \leq 3\delta, \quad \text{and} \quad \Big|\|w_j\|_{L^2}^2 - (1-\gamma)\Big| \leq 9\delta \tag{4.12}$$

holds for $j \geq j_0$. Next we are going to transfer these estimates for every $k \in \mathbb{N}$ to the functions obtained in an analogous way from the low-frequency parts $u_{j,k}^{(l)}$ of u_j, cf. (4.3). To this end, we introduce

$$v_{j,k}(x) = \rho(x-x_j)u_{j,k}^{(l)}(x) \quad \text{and} \quad w_{j,k}(x) = \theta(x-x_j)u_{j,k}^{(l)}(x).$$

Since ρ and θ attain their values in $[0,1]$, it follows from (4.6) that

$$\|v_{j,k}\|_{L^2} \leq \|u_{j,k}^{(l)}\|_{L^2} \leq 1 \quad \text{and} \quad \|w_{j,k}\|_{L^2} \leq \|u_{j,k}^{(l)}\|_{L^2} \leq 1. \tag{4.13}$$

Moreover, $v_{j,k} \in H^1$ and $w_{j,k} \in H^1$. Due to $\|\rho'\|_{L^\infty} \sim \delta \leq 1$ and $\|\theta'\|_{L^\infty} \sim \delta \leq 1$, we obtain, using (4.4), the bounds

$$\|v_{j,k}\|_{H^1} + \|w_{j,k}\|_{H^1} \leq CR_k. \tag{4.14}$$

The estimates from (4.12) are modified to

$$\|u_j - (v_{j,k} + w_{j,k})\|_{L^2} \leq 2(\delta^{1/2} + \varepsilon_k^{1/2}), \quad \Big|\|v_{j,k}\|_{L^2}^2 - \gamma\Big| \leq 3(\delta + \varepsilon_k^{1/2}), \tag{4.15}$$

$$\text{and} \quad \Big|\|w_{j,k}\|_{L^2}^2 - (1-\gamma)\Big| \leq 9(\delta + \varepsilon_k^{1/2}) \tag{4.16}$$

for $j \geq j_0$ and $k \in \mathbb{N}$. Indeed, since ρ attains values in $[0,1]$, (4.2) and (4.5) imply

$$\|v_j - v_{j,k}\|_{L^2}^2 = \int_{\mathbb{R}} \rho(x-x_j)^2 |u_j(x) - u_{j,k}^{(l)}(x)|^2\, dx \leq \|u_j - u_{j,k}^{(l)}\|_{L^2}^2 = \|u_{j,k}^{(h)}\|_{L^2}^2 \leq \varepsilon_k,$$

and in the same way $\|w_j - w_{j,k}\|_{L^2}^2 \leq \varepsilon_k$ follows, whence (4.15) and (4.16) are obtained. In particular, (4.15), (4.16), and (4.13) also yield

$$\Big|\|v_{j,k}\|_{L^2}^4 - \gamma^2\Big| \leq 6(\delta + \varepsilon_k^{1/2}), \quad \text{and} \quad \Big|\|w_{j,k}\|_{L^2}^4 - (1-\gamma)^2\Big| \leq 18(\delta + \varepsilon_k^{1/2}). \tag{4.17}$$

The next lemma derives a local bound on $|U(t)v_{j,k}|$ and $|U(t)w_{j,k}|$.

Lemma 4.1 *In the notation introduced above, we have*

$$\int_{|x-x_j|\geq x_2^* - 3\delta^{-1}} |U(t)v_{j,k}|^2\, dx \leq CR_k\delta \quad \textit{and} \quad \int_{|x-x_j|\leq x_2^* - 3\delta^{-1}} |U(t)w_{j,k}|^2\, dx \leq CR_k\delta,$$

for every $t \in [0,1]$, $j \geq j_0$ and $k \in \mathbb{N}$.

Proof: We can proceed similar to the proof of (2.18). To verify the estimate with $w_{j,k}$, we fix a function $\beta \in C_0^\infty(\mathbb{R})$ attaining values in $[0,1]$ such that $\beta(x) = 1$ for $|x| \le x_2^* - 3\delta^{-1}$, $\beta(x) = 0$ for $|x| \ge x_2^* - 2\delta^{-1}$, and $\|\beta'\|_{L^\infty} \le C\delta$. With $I(t) = \int_{\mathbb{R}} |(U(t)w_{j,k})(x)|^2 \beta(x - x_j)\, dx$, and writing $w_{j,k}(t) = U(t)w_{j,k}$, it follows from (1.2), (4.13), and (4.14) that

$$\begin{aligned} \dot I(t) &= (-2)\mathrm{Im} \int_{\mathbb{R}} \bar w_{j,k}(t)(\partial_x w_{j,k}(t))\beta'(x - x_j)\, dx \\ &\le C\|\beta'\|_{L^\infty} \|w_{j,k}(t)\|_{L^2} \|\partial_x w_{j,k}(t)\|_{L^2} \le C R_k \delta. \end{aligned}$$

Thus for $t \in [0,1]$ we obtain

$$\int_{|x-x_j|\le x_2^*-3\delta^{-1}} |U(t)w_{j,k}|^2\, dx = \int_{|x-x_j|\le x_2^*-3\delta^{-1}} |(U(t)w_{j,k})(x)|^2 \beta(x - x_j)\, dx \le I(t) \le I(0) + C R_k \delta.$$

But

$$I(0) = \int_{\mathbb{R}} |w_{j,k}(x)|^2 \beta(x - x_j)\, dx = \int_{|x-x_j|\le x_2^*-2\delta^{-1}} |u_{j,k}^{(l)}(x)|^2\, \theta(x - x_j)^2 \beta(x - x_j)\, dx = 0,$$

since $\theta(z) = 0$ for $|z| \le x_2^* - 2\delta^{-1}$, cf. (3) in Lemma 3.1. Concerning the estimate with $v_{j,k}$, one can argue in an analogous way by fixing a function $\beta \in C_0^\infty(\mathbb{R})$ which attains its values in $[0,1]$ such that $\beta(x) = 1$ for $|x| \ge x_1^* + 3\delta^{-1}$ and $\beta(x) = 0$ for $|x| \le x_1^* + 2\delta^{-1}$. Recalling $\rho(z) = 0$ for $|z| \ge x_1^* + 2\delta^{-1}$, it then follows that

$$\int_{|x-x_j|\ge x_1^*+3\delta^{-1}} |U(t)v_{j,k}|^2\, dx \le C R_k \delta,$$

and this yields the claimed estimated, as $x_2^* - x_1^* \ge 6\delta^{-1}$ implies $\int_{|x-x_j|\ge x_2^*-3\delta^{-1}} |U(t)v_{j,k}|^2\, dx \le \int_{|x-x_j|\ge x_1^*+3\delta^{-1}} |U(t)v_{j,k}|^2\, dx$. □

We now define $h_{j,k} = u_j - (v_{j,k} + w_{j,k})$ and observe that $\|h_{j,k}\|_{L^2} \le 2(\delta^{1/2} + \varepsilon_k^{1/2})$ by (4.15). Then we apply Lemma 2.9 and (4.13) to obtain the bound

$$\begin{aligned} &|\varphi(u_j) - \varphi(v_{j,k}) - \varphi(w_{j,k})| \\ &\quad\le C\Big(1 + \|u_j\|_{L^2}^3 + \|v_{j,k}\|_{L^2}^3 + \|w_{j,k}\|_{L^2}^3\Big)\|h_{j,k}\|_{L^2} \\ &\qquad + C\Big(|\Lambda_1(v_{j,k}, w_{j,k})| + |\Lambda_2(v_{j,k}, w_{j,k})| + |\Lambda_3(v_{j,k}, w_{j,k})| + \Lambda_4(v_{j,k}, w_{j,k})\Big) \\ &\quad\le C(\delta^{1/2} + \varepsilon_k^{1/2}) + C\Big(|\Lambda_1(v_{j,k}, w_{j,k})| + |\Lambda_2(v_{j,k}, w_{j,k})| + |\Lambda_3(v_{j,k}, w_{j,k})| + \Lambda_4(v_{j,k}, w_{j,k})\Big), \end{aligned} \tag{4.18}$$

with the remainder terms $\Lambda_1, \ldots, \Lambda_4$ as given in (2.20) and (2.21).

Lemma 4.2 *The estimate*

$$|\Lambda_1(v_{j,k}, w_{j,k})| + |\Lambda_2(v_{j,k}, w_{j,k})| + |\Lambda_3(v_{j,k}, w_{j,k})| + \Lambda_4(v_{j,k}, w_{j,k}) \le C R_k^{1/2} \delta^{1/2}$$

holds.

Proof: With $\Omega_1 = \{x \in \mathbb{R} : |x - x_j| \geq x_2^* - 3\delta^{-1}\}$ and $\Omega_2 = \{x \in \mathbb{R} : |x - x_j| \leq x_2^* - 3\delta^{-1}\}$, and moreover $u_1 = u_3 = v_{j,k}$, $u_2 = u_4 = w_{j,k}$, $i_1 = 1$, and $i_2 = 2$, it follows from Lemma 2.4, (4.13), and Lemma 4.1 that

$$\begin{aligned}
|\Lambda_1(v_{j,k}, w_{j,k})| &= \Big| \int_0^1 \int_{\mathbb{R}} (U(t)v_{j,k})^2 (\overline{U(t)w_{j,k}})^2 \, dxdt \Big| \\
&\leq C\Big(\|U(\cdot)v_{j,k}\|_{L^2([0,1]\times\Omega_1)} \|v_{j,k}\|_{L^2} \|w_{j,k}\|_{L^2}^2 + \|U(\cdot)w_{j,k}\|_{L^2([0,1]\times\Omega_2)} \|v_{j,k}\|_{L^2}^2 \|w_{j,k}\|_{L^2} \Big) \\
&\leq C\Big(\int_0^1 \int_{\Omega_1} |U(t)v_{j,k}|^2 \, dxdt \Big)^{1/2} + C\Big(\int_0^1 \int_{\Omega_2} |U(t)w_{j,k}|^2 \, dxdt \Big)^{1/2} \\
&\leq C R_k^{1/2} \delta^{1/2}.
\end{aligned}$$

The terms with Λ_2, Λ_3, and Λ_4 can be handled in the same way, the only important point is to note that each of these terms has at least one v-factor and at least one w-factor. Thus Lemma 2.4 can be applied to split the dx-integral and to integrate the respective factors over the sets on which they are small, by Lemma 4.1. □

Using Lemma 4.2, we can continue in (4.18) and obtain

$$|\varphi(u_j) - \varphi(v_{j,k}) - \varphi(w_{j,k})| \leq C(\delta^{1/2} + \varepsilon_k^{1/2}) + C R_k^{1/2} \delta^{1/2} \leq C(\varepsilon_k^{1/2} + R_k^{1/2} \delta^{1/2}). \tag{4.19}$$

This estimate holds for all $k \in \mathbb{N}$, $\delta \in]0, \gamma[$, and $j \geq j_0(\delta)$, with however $v_{j,k}$ and $w_{j,k}$ depending on δ, j, and k. Next, from (4.19), (2.11) in Lemma 2.6, and (4.17) it follows that

$$\begin{aligned}
\varphi(u_j) &\geq \varphi(v_{j,k}) + \varphi(w_{j,k}) - C(\varepsilon_k^{1/2} + R_k^{1/2}\delta^{1/2}) \\
&\geq \|v_{j,k}\|_{L^2}^4 P_1 + \|w_{j,k}\|_{L^2}^4 P_1 - C(\varepsilon_k^{1/2} + R_k^{1/2}\delta^{1/2}) \\
&\geq \gamma^2 P_1 - 6(\delta + \varepsilon_k^{1/2}) + (1-\gamma)^2 P_1 - 18(\delta + \varepsilon_k^{1/2}) - C(\varepsilon_k^{1/2} + R_k^{1/2}\delta^{1/2}) \\
&\geq (\gamma^2 + (1-\gamma)^2) P_1 - C(\varepsilon_k^{1/2} + R_k^{1/2}\delta^{1/2}).
\end{aligned}$$

Since we have got rid of the functions $v_{j,k}$ and $w_{j,k}$, we may take the limit $j \to \infty$ to find

$$P_1 \geq (\gamma^2 + (1-\gamma)^2) P_1 - C(\varepsilon_k^{1/2} + R_k^{1/2}\delta^{1/2}),$$

recalling that (u_j) is a minimizing sequence. This estimate is satisfied for all $\delta \in]0, \gamma[$ and all $k \in \mathbb{N}$. Hence we can pass successively to the limits first $\delta \to 0$ and then $k \to \infty$ to arrive at $P_1 \geq (\gamma^2 + (1-\gamma)^2) P_1$. Due to $P_1 < 0$, cf. Lemma 2.6, it follows that $\gamma(1-\gamma) \leq 0$, in contradiction to $\gamma \in]0, 1[$. Thus the case considered in the present Section 4.1.3 can also not occur.

4.2 Case (2): The Fourier transforms are vanishing

We consider the case that alternative (2) from Lemma 3.1 holds for $(\hat{u}_j)$, i.e., for every $A > 0$ we have

$$\lim_{j\to\infty} \hat{\Gamma}_j(A) = 0, \quad \text{where} \quad \hat{\Gamma}_j(A) := \sup_{\xi_0 \in \mathbb{R}} \int_{\xi_0 - A}^{\xi_0 + A} |\hat{u}_j|^2 \, d\xi. \tag{4.20}$$

According to Lemma 2.10, for every $A > \delta > 0$ and $j \in \mathbb{N}$ the estimate

$$\begin{aligned}
\int_{\mathbb{R}} |\hat{Q}(u_j)|^2 \, d\xi &\leq C\|u_j\|_{L^2}^6 (\delta + A^{-1/2}\delta^{-1/2}) + C\|u_j\|_{L^2}^5 \, A \, \hat{\Gamma}_j(A)^{1/2} \\
&= C(\delta + A^{-1/2}\delta^{-1/2}) + CA\,\hat{\Gamma}_j(A)^{1/2}
\end{aligned}$$

is satisfied. Given $\varepsilon > 0$ we choose $\delta = \varepsilon$ and fix $A = \varepsilon^{-3}$. With this A we apply (4.20) to find $j_0 \in \mathbb{N}$ such that $A\,\hat{\Gamma}_j(A)^{1/2} \le \varepsilon$ for $j \ge j_0$. Thus $\int_{\mathbb{R}} |\hat{Q}(u_j)|^2\,d\xi \le C\varepsilon$ for $j \ge j_0$ shows that in fact

$$\lim_{j\to\infty} \|Q(u_j)\|_{L^2} = \lim_{j\to\infty} \|\hat{Q}(u_j)\|_{L^2} = \lim_{j\to\infty} \Big(\int_{\mathbb{R}} |\hat{Q}(u_j)|^2\,d\xi\Big)^{1/2} = 0$$

is verified. Then we obtain from Lemma 2.2 that also $|\varphi(u_j)| = |(Q(u_j), u_j)_{L^2}| \le \|Q(u_j)\|_{L^2} \to 0$, i.e., $0 = \lim_{j\to\infty} \varphi(u_j) = P_1$, in contradiction to Lemma 2.6. Hence the case considered here can not occur.

4.3 Case (3): The Fourier transforms are splitting

In this section we suppose that alternative (3) from Lemma 3.1 is satisfied for $(\hat{u}_j)$. Then we have $\hat{\gamma} \in]0,1[$ for $\hat{\gamma} = \lim_{\xi\to\infty} \hat{\Gamma}(\xi)$, the function $\hat{\Gamma}$ being the pointwise limit (outside a countable set) of the concentration functions $\hat{\Gamma}_j(\xi) := \sup_{\xi_0\in\mathbb{R}} \int_{\xi_0-\xi}^{\xi_0+\xi} |\hat{u}_j|^2\,d\eta$ of the $\hat{u}_j$. We fix $\delta \in]0, \hat{\gamma}[$ and select $j_0 \in \mathbb{N}$, $\xi_1^* = z_1^* \in \mathbb{R}$, $\xi_2^* = z_2^* \in \mathbb{R}$, $\xi_j = z_j$ for $j \ge j_0$, and moreover the functions ρ and θ as stated in (3) of Lemma 3.1; once again, all these quantities are depending on δ. With $a_j(\xi) = \rho(\xi - \xi_j)\hat{u}_j(\xi)$ and $b_j(\xi) = \theta(\xi - \xi_j)\hat{u}_j(\xi)$, we then have $\|a_j\|_{L^2} \le 1$, $\|b_j\|_{L^2} \le 1$, and in addition for $j \ge j_0$ the estimates $\|\hat{u}_j - (a_j + b_j)\|_{L^2}^2 \le 2\delta$, $|\|a_j\|_{L^2}^2 - \hat{\gamma}| \le 3\delta$, as well as $|\|b_j\|_{L^2}^2 - (1-\hat{\gamma})| \le 9\delta$. Setting $v_j = \check{a}_j$ and $w_j = \check{b}_j$, this leads to $\|v_j\|_{L^2} \le 1$, $\|w_j\|_{L^2} \le 1$, and also

$$\|u_j - (v_j + w_j)\|_{L^2}^2 \le 2\delta, \quad \Big|\|v_j\|_{L^2}^2 - \hat{\gamma}\Big| \le 3\delta, \quad \text{and} \quad \Big|\|w_j\|_{L^2}^2 - (1-\hat{\gamma})\Big| \le 9\delta \tag{4.21}$$

for $j \ge j_0$. With $h_j = u_j - (v_j + w_j)$ then Lemma 2.9 implies

$$\begin{aligned} |\varphi(u_j) - \varphi(v_j) - \varphi(w_j)| &\le C\Big(1 + \|u_j\|_{L^2}^3 + \|v_j\|_{L^2}^3 + \|w_j\|_{L^2}^3\Big)\|h_j\|_{L^2} \\ &\quad + C\Big(|\Lambda_1(v_j,w_j)| + |\Lambda_2(v_j,w_j)| + |\Lambda_3(v_j,w_j)| + \Lambda_4(v_j,w_j)\Big) \\ &\le C\delta^{1/2} + C\Big(|\Lambda_1(v_j,w_j)| + |\Lambda_2(v_j,w_j)| + |\Lambda_3(v_j,w_j)| + \Lambda_4(v_j,w_j)\Big), \end{aligned} \tag{4.22}$$

the remainder terms $\Lambda_1, \ldots, \Lambda_4$ being defined in (2.20) and (2.21).

Lemma 4.3 *We have the bound*

$$|\Lambda_1(v_j,w_j)| + |\Lambda_2(v_j,w_j)| + |\Lambda_3(v_j,w_j)| + \Lambda_4(v_j,w_j) \le C\delta^{1/3}.$$

Proof: We are going to apply Lemma 2.11 with $v = v_j$, $w = w_j$, and $\xi_0 = \xi_j$. Note that the support assumptions on $\hat{v}$ and $\hat{w}$ are satisfied, since $\hat{v}_j(\xi) = a_j(\xi) = \rho(\xi - \xi_j)\hat{u}_j(\xi) = 0$ for $|\xi - \xi_j| \ge \xi_1^* + 2\delta^{-1}$, and also $\hat{w}_j(\xi) = b_j(\xi) = \theta(\xi - \xi_j)\hat{u}_j(\xi) = 0$ for $|\xi - \xi_j| \le \xi_2^* - 2\delta^{-1}$, see (3) in Lemma 3.1. Due to

$$\Lambda_1(v_j,w_j) = \int_0^1 \int_{\mathbb{R}} (U(t)v_j)^2 (\overline{U(t)w_j})^2\,dxdt = \int_{\mathbb{R}} Q(v_j,w_j,v_j)\overline{w_j}\,dx,$$

cf. (2.20), (1.8), and (2.7), hence (2.27) in Lemma 2.11 yields $|\Lambda_1(v_j,w_j)| \le C\|v_j\|_{L^2}^2 \|w_j\|_{L^2}^2 \delta^{1/3} \le C\delta^{1/3}$. Concerning Λ_2, here we have

$$\Lambda_2(v_j,w_j) = \int_0^1 \int_{\mathbb{R}} |U(t)v_j|^2 (U(t)v_j)(\overline{U(t)w_j})\,dxdt = \int_{\mathbb{R}} Q(v_j,v_j,v_j)\overline{w_j}\,dx,$$

thus (2.28) in Lemma 2.11 results in $|\Lambda_2(v_j,w_j)| \leq C\delta^{1/3}$. For Λ_3 we note that

$$\Lambda_3(v_j,w_j) = \int_0^1 \int_{\mathbb{R}} |U(t)w_j|^2 (U(t)v_j)(\overline{U(t)w_j})\,dxdt = \int_{\mathbb{R}} Q(v_j,w_j,w_j)\overline{w_j}\,dx.$$

Therefore we can apply (2.29) from Lemma 2.11 to see that also $|\Lambda_3(v_j,w_j)| \leq C\delta^{1/3}$. Finally, to bound Λ_4 we have

$$\Lambda_4(v_j,w_j) = \int_0^1 \int_{\mathbb{R}} |U(t)v_j|^2 |U(t)w_j|^2\,dxdt = \int_{\mathbb{R}} Q(v_j,v_j,w_j)\overline{w_j}\,dx.$$

Then (2.30) in Lemma 2.11 leads to $\Lambda_4(v_j,w_j) \leq C\delta^{1/3}$. □

Using Lemma 4.3 in (4.22), we thus obtain

$$|\varphi(u_j) - \varphi(v_j) - \varphi(w_j)| \leq C\delta^{1/2} + C\delta^{1/3} \leq C\delta^{1/3}, \quad j \geq j_0. \tag{4.23}$$

Due to this estimate we may now proceed analogously to Section 4.1.3, cf. (4.19). From (4.23), (2.11) in Lemma 2.6, and (4.21) it follows that

$$\begin{aligned} \varphi(u_j) &\geq \varphi(v_j) + \varphi(w_j) - C\delta^{1/3} \geq \|v_j\|_{L^2}^4 P_1 + \|w_j\|_{L^2}^4 P_1 - C\delta^{1/3} \\ &\geq \hat{\gamma}^2 P_1 - 6\delta|P_1| + (1-\hat{\gamma})^2 P_1 - 18\delta|P_1| - C\delta^{1/3} \\ &\geq \hat{\gamma}^2 P_1 + (1-\hat{\gamma})^2 P_1 - C\delta^{1/3} \end{aligned}$$

for $j \geq j_0$. Since (u_j) is a minimizing sequence, as $j \to \infty$ this yields $P_1 \geq \hat{\gamma}^2 P_1 + (1-\hat{\gamma})^2 P_1 - C\delta^{1/3}$. Taking the limit $\delta \to 0$ and recalling $P_1 < 0$, we finally arrive at $\hat{\gamma}(1-\hat{\gamma}) \leq 0$, contradicting $\hat{\gamma} \in]0,1[$. Hence the case considered here is not possible.

4.4 Summary and conclusion of the proof of Theorem 1.1

At the beginning of Section 4 we have divided the argument into three cases, according to which one of the possibilities (1), (2), or (3) of Lemma 3.1 applied to $(\hat{u}_j)$ occurs. We have seen in Sections 4.2 and 4.3 that neither (2) nor (3) can hold, hence (1) is satisfied. Then the argument has been split further in Sections 4.1.1-4.1.3, depending on which alternative (1), (2), or (3) from Lemma 3.1 holds for (u_j). It has turned out that cases (2) and (3) are impossible, whence (1) is verified both for $(\hat{u}_j)$ and for (u_j). In this single case we have shown in Section 4.1.1 that the minimizing sequence has a strong L^2-limit, which is a solution of P_1. (We note that in fact it is a subsequence of $v_j(x) = e^{-i\xi_j(x_j+x)}u_j(x_j+x)$, for suitable ξ_j and x_j, which has a strong L^2-limit.) Hence there is a minimizer $u \in L^2$ for P_1. By the Lagrange multiplier rule we then have $-4Q(u) = 2\mu u$, where $\mu \in \mathbb{R}$ denotes the Lagrange multiplier. Taking the inner product with u, it follows that $\mu = 2\varphi(u) = 2P_1$, thus $u = -P_1^{-1}Q(u) \in L^2 \cap L^\infty$, cf. Lemma 2.12. This completes the proof of Theorem 1.1. □

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