1 Rigorous methods and numerical results for dry friction problems (M. Kunze)

It is the aim of the present section to first explain some mathematically rigorous methods that can be used to analyze dry friction problems, or non-smooth systems in general. The first concept, Lyapunov exponents, is well-accepted for smooth systems to predict the long-term behaviour, and the existence of at least one positive Lyapunov exponent is often used as the definition of "chaos". Then in section 1.2 we will turn to a different issue, namely to verify that some system has undergone a bifurcation when some external parameter has been varied. Often a good indicator for bifurcations appearing is that some "index" changes when the parameter crosses a critical value. We will introduce the so-called Conley index and explain how it applies to detect such bifurcations. Since usually the dynamical behaviour will be quite complicated, often only numerical analysis will yield satisfactory results. In section 1.3 we will describe some phenomena observed in a single-mass friction oscillator which are due to the fact that trajectories can collide with discontinuity surfaces. Most of the material is taken from [M. Kunze (1998b)].

1.1 Lyapunov exponents for non-smooth systems

We start with some general remarks concerning Lyapunov exponents. The idea of Lyapunov exponents is to define characteristic numbers for a dynamical system that allow to classify the behaviour of the system is a concise manner. These numbers should account for exponential convergence or divergence of trajectories that start close to each other. To give an example, consider the linear system

$$\dot{x} = Ax \tag{1}$$

in $\mathbf{R}^n$ with a constant $(n\times n)$-matrix A. It is known that the eigenvalues $\lambda_1,\ldots,\lambda_n$ of A play the crucial role for the long-term behaviour: If all real parts of the eigenvalues are less that zero, every solutions will tend to the fixed-point $x=0$ exponentially fast, whereas solutions are exponentially divergent in as many directions as there are positive real parts. To generalize this to non-linear systems

$$\dot{x} = f(t,x), \tag{2}$$

consider a reference solution trajectory $x(t)$ emerging from some initial value $x_0\in\mathbf{R}^n$. Then we perturb x_0 a little to some nearby $\bar{x}_0\in\mathbf{R}^n$ with solution $\bar{x}(t)$ emerging from $\bar{x}_0$. We are interested in the time evolution of the initial difference $z_0=\bar{x}_0-x_0$ and write for this purpose

$$\bar{x}(t)-x(t)\approx\frac{\partial\varphi}{\partial x}(t,x_0)(\bar{x}_0-x_0)=\frac{\partial\varphi}{\partial x}(t,x_0)z_0, \tag{3}$$

where $y(t)=\varphi(t,y_0)$ is the solution flow to (2) with initial value y_0, i.e., $x(t)=\varphi(t,x_0)$ and $\bar{x}(t)=\varphi(t,\bar{x}_0)$. In (3), $\frac{\partial\varphi}{\partial x}(t,x_0)$ if the Jakobian $(n\times n)$-matrix of the flow, taken w.r. to the initial value. Hence the quantity

$$\lambda^+(x_0,z_0):=\limsup_{t\to\infty}\left(\frac{1}{t}\log\left|\frac{\partial\varphi}{\partial x}(t,x_0)z_0\right|\right) \tag{4}$$

somehow describes the evolution of small perturbations of the initial value, and it is called an upper Lyapunov exponent; "upper" refers to the lim sup. On first sight it might appear as a mathematician's subtleness to write "$\limsup_{t\to\infty}$" instead of "$\lim_{t\to\infty}$", but at the moment nothing ensures us that the quantity in $(\dots)$ really approaches a definite value at $t\to\infty$ and not only oscillates between certain different values. We will return to this important point later. To see what this definition (4) gives for the linear example (1), we recall that the flow in that case may be written as $\varphi(t,y_0)=e^{tA}y_0$, with the $(n\times n)$-matrix e^{tA}. By transformation of A to Jordan's canonical form it may then be shown that the values of $\lambda^+(x_0,z_0)$ with x_0,z_0 ranging over $\mathbf{R}^n$ are exactly the real parts of the eigenvalues $\lambda_1,\dots,\lambda_n$ of A. Moreover, all the $\limsup_{t\to\infty}$ do exist as $\lim_{t\to\infty}$. This fact and the connection to the eigenvalues of A strongly indicates that the (upper) Lyapunov exponents defined by (4) will provide a useful tool for making predictions on the stability of the system. The existence of a positive exponent will indicate exponentially fast divergence of nearby trajectories, an issue that is often addressed as "chaos". Let us note that also for non-linear systems it can be shown that the $\lambda^+(x_0,z_0)$ with x_0,z_0 ranging over $\mathbf{R}^n$ do take at most n distinct values, comprising the "Lyapunov spectrum".

Now the problem is when these number can be used to make reliable predictions. This is an important point, since the Lyapunov spectrum is also accessible through numerical computations, even for quite complex systems. On the theoretical side, of course the $\limsup_{t\to\infty}$ has to exist as a $\lim_{t\to\infty}$, so we need a criterion for that to hold. Such criterion is provided by the famous ergodic theorem of Oseledets [V.I. Oseledets (1968)] that relies on some *a priori* probability measure μ, being naturally related to the system and invariant for the flow, i.e., under the flow the μ-measure of sets in phase space remains constant. Additionally, μ is assumed to be ergodic, meaning that every flow-invariant set has μ-measure either zero or one. The theorem reads as follows.

Theorem. *Let μ be an ergodic probability measure which is invariant w.r. to the flow. Then there exists a set $\Gamma\subset\mathbf{R}^n$ of full μ-measure and $k\le n$ distinct numbers $\lambda^{(1)},\dots,\lambda^{(k)}\in\mathbf{R}$ such that for all initial points $x_0\in\Gamma$ and all $z_0\in\mathbf{R}^n$ the quantity $\lambda^+(x_0,z_0)$ from (4) equals one of the $\lambda^{(i)}$, and additionally the* $\limsup_{t\to\infty}$ *is in fact the* $\lim_{t\to\infty}$.

So we are now already in a good position, since we know when the Lyapunov exponents exist. Nevertheless, note that the assertion of the theorem is intimately connected to μ; if μ is a Dirac mass at some fixed point x_0, then from the theorem we can conclude the existence of $\lim_{t\to\infty}$ in (4) only for this particular x_0, since then necessarily $\Gamma=\{x_0\}$.

To conclude this brief introduction to Lyapunov exponents, we recall that they can equivalently be defined for dynamical systems generated by the iteration of maps $f:\mathbf{R}^n\to\mathbf{R}^n$, instead of dynamical systems generated by flows. In that case one lets

$$\lambda^+(x_0,z_0)=\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\left|\frac{df}{dx}(f^i(x_0))z_0\right|,$$

with f^i being the i'th iterate of f. Perhaps the most famous example of this is the parameter-dependent one-dimensional logistic map $f(x)=ax(1-x)$ on $[0,1]$, where $a\in[0,4]$. Again the Lyapunov exponents exist as a $\lim_{n\to\infty}$ instead of the $\limsup_{n\to\infty}$, and also much is known about the corresponding invariant measures. Moreover, the exponents clearly reflect the well-known chaotic regimes and the sequence of period doublings leading to the first chaotic regime; cf. [W. de Melo and S. van Strien (1993)].

Now to see what the difficulty is with Lyapunov exponents for non-smooth systems, we consider the example

$$\ddot{x}+x+\mathrm{sgn}(\dot{x})=\gamma\sin(\eta t). \tag{5}$$

Figure 1: A trajectory hits some discontinuity surface D tangentially

This is one of the simplest dry-friction models, describing a pendulum that consists of a mass being attached to a spring, and the mass, which is sinusoidally forced, moves in a straight tube and has contact to the wall of the tube. Depending on the size of the dry friction between mass and wall, and depending on the strength of the forcing, the mass moves up or down ("slip phase") or it sticks to the wall. The term $\mathrm{sgn}(\dot{x})$ accounts for the dry friction, by Coulomb's friction law, since the friction force is proportional to the negative of the normal force (constant, since the tube is straight) multiplied by the sign of the velocity. In realistic applications one would certainly use a more refined friction characteristics. Note, too, that in (5) we have set all physical constants equal to one, besides amplitude γ and frequency $(2\pi)/\eta$ of the forcing.

Solution trajectories of (5) pass either through $\{\dot{x}>0\}$ or through $\{\dot{x}<0\}$, where we just have a simple oscillator. However, there is a third possibility that makes (5) non-linear, namely a solution trajectory can stick to $\{\dot{x}=0\}$ for some time, corresponding to sticking of the mass to the wall. In that case we have $x(t)\equiv x_0$ in some time interval $t\in[t_1,t_2]$. Hence from (5) we find that for sticking to happen we need to have $x_0+\mathrm{sgn}(0)=\gamma\sin(\eta t)$ for $t\in[t_1,t_2]$, and this is impossible if we would define $\mathrm{sgn}(0):=0$. The proper idea is to take $\mathrm{sgn}(0)$ as the whole interval $[-1,1]$ that "connects" $+1$ to -1. Therefore we arrive at the condition $x_0\in[\gamma\sin(\eta t)-1,\gamma\sin(\eta t)+1]$ for $t\in[t_1,t_2]$. From this criterion we can decide whether a solution that hits the surface $\{\dot{x}=0\}$ in phase space at time t_1 will just pass from one to the other side, or whether it has to rest up to some time $t_2>t_1$. Note that for amplitudes $\gamma\leq 1$ there are some solutions sticking forever ($t_2=\infty$), e.g. the one with $x_0=0$.

The mathematical theory of such differential equations where certain functions take as values intervals (like here $\mathrm{sgn}(0):=[-1,1]$), or even more general sets, is well-established, see [J.P. Aubin and A. Cellina (1984), K. Deimling (1992)]. From these results we know that (5) defines a semi-flow, i.e., a flow for positive times. (Note that we cannot expect solutions to be unique in backward time, since the discontinuity surface can be left at will in backward time.) We denote this semi-flow by $\varphi(t,y_0)$ $(t\geq 0)$ where, as (5) is non-autonomous, the initial value $y_0=(x_0,\dot{x}_0,t_0)$ has three components: Some starting time t_0, and the position x_0 and velocity $\dot{x}_0$ at time t_0, respectively.

Our general goal is to define and estimate Lyapunov exponents for systems like (5), and it is already here that we encounter the first main problem. As (5) is non-smooth, the semi-flow will not be smooth w.r. to the initial value y_0, and the corresponding $\lambda^+(y_0,z_0)$ from (4) is not well-defined, since it would require $\frac{\partial\varphi}{\partial y}(t,y_0)$ to exists. E.g. think of a trajectory that touches upon the discontinuity surface D tangentially ("grazing"), with a nearby initial value $\bar{y}_0$ leading to a trajectory that just crosses from, say, $\{\dot{x}>0\}$ to $\{\dot{x}<0\}$; see Fig. 1. However, this phenomenon is somehow of a lower-dimensional nature.
Nevertheless, $\frac{\partial\varphi}{\partial y}(t,y_0)$ is not defined at some points y_0, as would be necessary to apply Oseledets' theorem in the above version in order to have a well-defined Lyapunov spectrum. Here it helps that the theorem in fact is much more general as stated. Instead of the linearization it is only necessary to have what is called a cocyle, i.e., a map $T:[0,\infty[\times\mathbf{R}^n\rightarrow$

Figure 2: Derivation of the transition matrix for the linearization

$M(n\times n)$ (the $(n\times n)$-matrices) which satisfies

$$T(t+s,y_0)=T(t,\varphi(s,y_0))\circ T(s,y_0) \quad \text{for} \quad t,s\in[0,\infty[\quad \text{and} \quad y_0\in\mathbf{R}^n. \tag{6}$$

The corresponding generalization of (4) is

$$\lambda^+(y_0,z_0):=\limsup_{t\to\infty}\left(\frac{1}{t}\log|T(t,y_0)z_0|\right), \tag{7}$$

and the ergodic theorem of Oseledets then holds accordingly. Observe that the identity in (6) follows from the flow equation $\varphi(t+s,y_0)=\varphi(t,\varphi(s,y_0))$ in the smooth case by differentiation w.r. to the initial value y_0.

Hence our problem has been shifted to define a suitable cocycle for a non-smooth system as (5). Of course we will set $T(t,y_0)=\frac{\partial\varphi}{\partial y}(t,y_0)$ where this is possible, i.e., while the trajectory stays in either $\{\dot{x}>0\}$ or $\{\dot{x}<0\}$. We certainly will have to exclude from the definition of $T(t,y_0)$ initial values y_0 like above that lead to grazing, and also infinitely many hits upon the discontinuity surface in a finite time poses problems; this occurs e.g. with a gyroscopic pendulum that can have contact to a wall while rotating. However, it can be rigorously proved that it does not happen for examples like (5), and it can also be shown that the set of initial values which will *not* lead to grazing at some later time do form an invariant set G ("good initial values") of full Lebesgue measure in phase space. Therefore it is reasonable to define $T(t,y_0)$ only for $y_0\in G$, and it is enough to clarify what happens at some switching time from one of the regions $\{\dot{x}>0\}$, $\{\dot{x}<0\}$, or the "sticking region", to another, with the trajectory arriving non-tangentially. Therefore we have to derive a formula that describes how the linearization of the semi-flow is transformed while crossing a certain discontinuity surface. Following [M.A. Aizerman and F.R. Gantmakher (1958), S.F. Bockman (1991), M. Kunze (1998b), M. Kunze and B. Michaeli (1995), P.C. Müller (1995)], the formula is obtained as follows. We consider an initial point $y_0\in G$ that leads to consecutive switching times $0=t_0(y_0)<t_1(y_0)<t_2(y_0)<\ldots$ from one of the regions to another. Assume that at time $t_{i+1}=t_{i+1}(y_0)$ and in the point $\xi_{i+1}=\xi_{i+1}(y_0)$ the trajectory hits some discontinuity surface $\mathcal{D}$ which can be described as the level set $\{y:h_l(y)=0\}$ for a smooth function h_l. Moreover, we suppose that just before the hitting the governing equation is $\dot{y}=f_l(y)$ with flow φ_l, while shortly afterwards it becomes $\dot{y}=f_p(y)$ with flow φ_p; see Fig. 2. We will derive the form of the transition matrix $A_{i+1}=A_{i+1}(y_0)$ of the linearization, i.e., A_{i+1} will satisfy

$$\lim_{t\to t_{i+1}^+}\frac{\partial\varphi}{\partial y}(t,y_0)=A_{i+1}\circ\left(\lim_{t\to t_{i+1}^-}\frac{\partial\varphi}{\partial y}(t,y_0)\right). \tag{8}$$

To get (8), note $\varphi(t,y_0)=\varphi_p(t-t_{i+1},\xi_{i+1})$ for $t\in[t_{i+1},t_{i+2}[$, hence

$$\begin{aligned}
&\frac{\partial\varphi}{\partial y}(t,y_0)\\
&=-\frac{\partial\varphi_p}{\partial t}(t-t_{i+1},\xi_{i+1})\frac{dt_{i+1}}{dy}(y_0)+\frac{\partial\varphi_p}{\partial y}(t-t_{i+1},\xi_{i+1})\frac{d\xi_{i+1}}{dy}(y_0)\\
&=-f_p(\varphi_p(t-t_{i+1},\xi_{i+1}))\frac{dt_{i+1}}{dy}(y_0)+\frac{\partial\varphi_p}{\partial y}(t-t_{i+1},\xi_{i+1})\frac{d\xi_{i+1}}{dy}(y_0).
\end{aligned}$$

Therefore

$$\lim_{t\to t_{i+1}^{+}} \frac{\partial\varphi}{\partial y}(t,y_0) = -f_p(\xi_{i+1})\frac{dt_{i+1}}{dy}(y_0) + \frac{d\xi_{i+1}}{dy}(y_0). \tag{9}$$

Denote $\Gamma(t,y)=\varphi_l(t-t_i(y),\xi_i(y))$. Then $\Gamma(t_{i+1}(y),y)=\xi_{i+1}(y)$, thus

$$\frac{d\xi_{i+1}}{dy}(y_0) = f_l(\xi_{i+1})\frac{dt_{i+1}}{dy}(y_0) + \frac{\partial\Gamma}{\partial y}(t_{i+1},y_0). \tag{10}$$

By differentiation of $0=h_l(\xi_{i+1}(y))=h_l(\Gamma(t_{i+1}(y),y))$ we additionally obtain

$$\frac{dt_{i+1}}{dy}(y_0) = -\frac{1}{\langle\nabla h_l, f_l\rangle(\xi_{i+1})}\langle\nabla h_l(\xi_{i+1}), \frac{\partial\Gamma}{\partial y}(t_{i+1},y_0)\cdot\rangle. \tag{11}$$

As $\frac{\partial\varphi}{\partial y}(t,y_0)\to\frac{\partial\Gamma}{\partial y}(t_{i+1},y_0)$ for $t\to t_{i+1}^{-}$, (8) follows from (9), (10), and (11), with the matrix

$$\begin{aligned} A_{i+1} := {} & \frac{1}{\langle\nabla h_l, f_l\rangle(\xi_{i+1})}\Big([f_p(\xi_{i+1})-f_l(\xi_{i+1})]_\alpha\cdot[\nabla h_l(\xi_{i+1})]_\beta\Big)_{1\le\alpha,\beta\le n} \\ & +\mathrm{id}_{\mathbf{R}^n} \end{aligned} \tag{12}$$

for a general problem in $\mathbf{R}^n$, $[\ldots]_\alpha$ denoting the α's component of a n-vector.

As an example, we consider (5) and a switching from $\{\dot{x}>0\}$ to $\{\dot{x}<0\}$ at time $t_{i+1}=t_{i+1}(y_0)$, for a trajectory starting at some $y_0=(x_0,\dot{x}_0,t_0)$. With $\xi_{i+1}:=(x_{i+1},\dot{x}_{i+1},t_{i+1})$, $h_l(x,v,t)=v$, $f_p(x,v,t)=(v,-x+\gamma\sin(\eta t)+1,1)$ [the "1" in the third component due to $\dot{t}=1$ for (5) written as a time-autonomous system], $f_l(x,v,t)=(v,-x+\gamma\sin(\eta t)-1,1)$, it results that here

$$A_{i+1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta_{i+1}^{+} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \theta_{i+1}^{+} = 1-\frac{2}{x_{i+1}-\gamma\sin(\eta t_{i+1})+1}. \tag{13}$$

Returning to the general set-up, we can define

$$T(t,y_0) = \begin{cases} \frac{\partial\varphi}{\partial y}(t,y_0) & : \ y_0\in G, \\ & \ \ t \text{ is no switching time} \\ A_{i+1}(y_0)\Big(\lim\limits_{s\to t_{i+1}(y_0)^-} T(s,y_0)\Big) & : \ y_0\in G,\, t=t_{i+1}(y_0) \\ 0 & : \ y_0\notin G \end{cases}$$

and it then can be verified that $T(t,y_0)$ is a cocycle, in the sense that (6) holds. Hence we have shown

Lemma 1 *For systems like (5), a canonical cocycle can be defined. Hence Oseledets' ergodic theorem applies (if there is a suitable invariant ergodic measure), and it ensures that the corresponding Lyapunov spectrum is well-defined.*

See [M. Kunze (1998b)] for full details.

A word of warning is in order here, also related to our insisting that the $\limsup_{t\to\infty}$ in (7) or (4) should be a $\lim_{t\to\infty}$ to allow reliable predictions. The situation for (5) is completely clarified in the resonant case $\eta=1$; see [K. Deimling (1994), K. Deimling and P. Szilágyi (1994), M. Kunze (1998b), M. Kunze (1998a)].

(a) For $\gamma \in [0,1]$, the equilibria $x_0 \in [\gamma-1, 1-\gamma]$ are the only 2π-periodic solutions. All solutions of (5) are asymptotically attracted to this interval $[\gamma-1, 1-\gamma]$ in the $(x,\dot{x})$-phase plane.

(b) For $\gamma \in]1, 4/\pi[$, there exists a unique 2π-periodic solution. The pendulum sticks to the wall twice during the period interval, and the solution asymptotically attracts all other solutions of (5).

(c) For $\gamma = 4/\pi$, there are infinitely many 2π-periodic solutions of (5); they are explicitly known.

(d) For $\gamma \in]4/\pi, \infty[$, *all* solutions of (5) are unbounded in the $(x,\dot{x})$-phase plane.

This behaviour is well-reflected through the numerically obtained Lyapunov spectrum for $\gamma \in [0, 4/\pi[$; see [B. Michaeli (1998)]. However, for $\gamma > 4/\pi$ the exponents seem to be almost zero, but this is false by (d). From a mathematical viewpoint, the solutions being unbounded will permit an invariant measure for the system to exist, since the flow will enlarge volume. Hence without an invariant measure the ergodic theorem cannot be applied, and therefore it makes no sense to talk about Lyapunov exponents at all.

With Lemma 1 at hand, it will be the next task to estimate the Lyapunov exponents in mathematically accessible cases as (5). Here we have the following theorem.

Theorem 1 *For all $y_0, z_0 \in \mathbf{R}^3$,*

$$\lambda^+(y_0, z_0) \leq 0.$$

In particular, the pendulum with dry friction (5) has no positive Lyapunov exponent. This holds for every choice of amplitude $\gamma \geq 0$ and frequency $(2\pi)/\eta \geq 0$.

The proof of this result goes as follows. By (7) and due to the sublinearity properties of the exponents, it will be enough to estimate

$$|T(t,y_0)e_j|, \quad y_0 \in \mathbf{R}^3, \quad j = 1,2,3,$$

with the unit vectors e_j in $\mathbf{R}^3$. By the form of T we only need to consider "good" initial values $y_0 \in G$. Such y_0 defines a sequence of switching times $0 = t_0 < t_1 < t_2 < \ldots$ as above, and we have $t_i \to \infty$, since there is no accumulation point of switching times. At each t_{i+1} the cocycle is multiplied by the corresponding transition matrix A_{i+1}, cf. (12), and it can be shown by analyzing all possible cases of switchings that each A_{i+1} will be of the form (13), where possibly θ^+_{i+1} is replaced by

$$\theta^-_{i+1} = 1 - \frac{2}{\gamma \sin(\eta t_{i+1}) + 1 - x_{i+1}}$$

or by $\theta^0_{i+1} = 0$, the latter in case the trajectory switches to the sticking region. Then also $\theta^+_{i+1} \in [0,1]$ and $\theta^-_{i+1} \in [0,1]$, as is obtained by an analysis of the vector field. Hence all the transition matrices like (13) are of norm less or equal to unity. Note that this nicely reflects the dissipative nature of (5), and the largest dissipation occurs when the pendulum has to stick to the wall for some time.

To estimate $|T(t,y_0)e_j|$, according to the definition of T we have

$$T(t,y_0) = B_{i+1}(t) \circ \Big(\lim_{s \to t^-_{i+1}} T(s,y_0) \Big)$$

for $t\in[t_{i+1},t_{i+2}[$ with a suitable matrix $B_{i+1}(t)$ coming from the linearization of the semi-flow. This may then be iterated to yield for integers i and $t\in[t_{i+1},t_{i+2}[$

$$T(t,y_0)=B_{i+1}(t)\circ B_i(t_{i+1})\circ\ldots\circ B_1(t_2)\circ T_1^- \tag{14}$$

with $T_1^-=\lim_{s\to t_1^-}T(s,y_0)$. The proper estimation of (14) is still quite technical, but it can be shown, mainly since $\theta_{i+1}^+\in[0,1]$ and $\theta_{i+1}^-\in[0,1]$, that with each additional multiplication of a matrix $B_{i+1}(t)$ the norm in (14) can increase at most at $\approx(t-t_{i+1})$; see [M. Kunze (1998b)]. Hence we obtain the bound

$$|T(t,y_0)e_j|\approx\Big((t-t_{i+1})+\sum_{k=0}^{i}(t_{k+1}-t_k)\Big)=t,$$

and therefore by (7),

$$\lambda^+(y_0,e_j)\le\limsup_{t\to\infty}\left(\frac{\log t}{t}\right)=0.$$

This concludes our short sketch of the proof of Theorem 1.

Finally we remark that although (5) is non-linear, the equation may be considered as "almost" linear, but recall that the switching times from one "region of linearity" to another depend on the initial value. Thus Theorem 1 may not come as a surprise, but we want to emphasize that it is a mathematically rigorous result. Even for simple two-dimensional iterated maps like the Henon map $f:\mathbf{R}^2\to\mathbf{R}^2$, $f(x,y)=(1-ax^2+y,bx)$, with parameters $a,b\ge 0$ it is already extremely difficult to identify those a and b which lead to positive Lyapunov exponents; cf. [L.-S. Young (1998)]. Numerically, positive Lyapunov exponents are observed for models like (5) with non-linear friction characteristics that behave e.g. as v^2 for $v\to\infty$.

1.2 Application of Conley index theory to non-smooth systems

In this section we want to explain a powerful mathematical method to prove bifurcations for non-smooth systems depending on some bifurcation parameter. As before, we start with a general introduction to the subject, following in part [C. Conley (1978), Ch. I.2], [J. Smoller (1994), Sec. 22.A], or [M. Kunze (1998b), Sec. 4]. See the latter for more references on the subject.

It was mentioned in the introduction that, generally speaking, the change of some kind of index usually is a signal for a bifurcation in a non-linear dynamical system $\dot{x}=f_\lambda(x)$ depending on some parameter λ. Since we are interested in the properties of the flow, naturally the index should be connected to the changes of the flow. As an example, we consider

$$\dot{x}=f_\lambda(x)=x(1-x^2)-\lambda,\quad x\in\mathbf{R}. \tag{15}$$

For $\lambda=0$ there are three equilibria $x_0=-1,0,1$. Moreover, there exists a unique $\lambda_1>0$ such that there are exactly two equilibria $a_1=a(\lambda_1)<-1$ and $r_1=r(\lambda_1)\in]0,1[$, and for $\lambda>\lambda_1$ there is left only one equilibrium $a(\lambda)$ which then has $a(\lambda)<a_1<-1$; see Fig. 3.
The corresponding flows are indicated by arrows: for $\lambda=0$, -1 and 1 are stable while zero is unstable. Moreover, for $\lambda=\lambda_1$, a_1 is stable and r_1 is degenerated (since $f'_{\lambda_1}(r_1)=0$), and finally $a(\lambda)$ is stable for $\lambda>\lambda_1$.

Figure 3: Sketch of the flows for (15) on different λ-levels

It was now the idea of Conley to assign some index to special sets as the equilibria, and the index will account for changes in the phase portrait when λ is varied. The key notions he utilized are "isolating neighborhoods" and "isolated invariant sets". An isolating neighborhood is simply a set such that no boundary point stays in the set forever (in forward and backward time). There are plenty of isolating neighborhoods for the above example. In particular, every interval having no equilibrium as an endpoint will serve. Equivalently, an isolating neighborhood is a set which contains its maximal invariant subset in the interior. An isolated invariant set is then determined through the requirement that it is the maximal invariant subset of some isolating neighborhood. To see what this means, consider $\lambda = \lambda_1$. There are four isolated invariant sets, namely $\{a_1\}$, $\{r_1\}$, $\{a_1, r_1\}$, and $[a_1, r_1]$. E.g. $[a_1 - 1, r_1 + 1]$ is an isolating neighborhood, and the maximal invariant subset of $[a_1 - 1, r_1 + 1]$ (i.e. the points that never will leave the interval neither in forward nor in backward time) is $[a_1, r_1]$; this interval is contained in the interior of $[a_1 - 1, r_1 + 1]$. Note that two different isolating neighborhoods can give rise to the same isolated invariant set. For instance, we also have $\mathrm{inv}([a_1 - 2, r_1 + 2]) = [a_1, r_1]$, with $I = \mathrm{inv}(A)$ denoting the maximal invariant subset of A. Note also that all these definitions depend on the underlying flow, induced by $\dot{x} = f_\lambda(x)$.

To define the Conley index of an isolated invariant set, suppose now that for a given isolated invariant set I there exists a special isolating neighborhood B (called an isolating block) such that each boundary point leaves B immediately either in forward or in backward time, and denote b^+ the set of all boundary points that leave B immediately in forward time. Then we can determine the homotopy type of the quotient B/b^+ and define an index as $h(I) = [B/b^+]$, this homotopy type. The reader not familiar with such concepts need not worry, as this only amounts to identifying all points in b^+; they are collapsed to a single distinguished point. It can be shown that in fact an isolating block exists, and, moreover, that the index does not depend on the special choice of isolating block, i.e., if B_1, B_2 are isolated blocks with exit sets b_1^+, b_2^+ and $\mathrm{inv}(B_1) = I = \mathrm{inv}(B_2)$, then $[B_1/b_1^+] = [B_2/b_2^+]$.

Let us again illustrate this with the above example. First, consider the isolating block $B = [-1/2, 1/2]$ for the isolated invariant set $I = \{0\}$ on the level $\lambda = 0$. Then the immediate exit set is $b^+ = \{-1/2, 1/2\}$, so in order to find the homotopy type of B/b^+, both boundary points have to be identified to a new distinguished point, which then has to be attached to the set. By identifying $-1/2$ and $1/2$, B becomes a (pointed) circle, i.e., the (pointed) unit sphere in $\mathbf{R}^2$, and this homotopy type usually is denoted by Σ^1; "pointed" just means that one point of the sphere is distinguished as the one where both ends have been glued together. Thus $h(\{0\}; f_0) = \Sigma^1$; we explicitly indicated the right-hand side that induces the flow. Next consider the isolating block N for $\lambda \in [0, \lambda_2]$ as shown in Fig. 3. Here the exit set n^+ is empty, and therefore $N/n^+ \sim N \cup \{*\} \sim \{p, *\}$ (with $*$ being the distinguished point), since N can be contracted to a single point p. Thus $h(\mathrm{inv}(N); f_\lambda) = h(\{a(\lambda)\}; f_\lambda) = \Sigma^0$ is the (pointed) unit sphere $\{-1, 1\}$ in $\mathbf{R}$, i.e., the (pointed) two-point space, for $\lambda \in [0, \lambda_2]$. As a last example we will deal with $\tilde{N}$, see Fig. 3, hence $\mathrm{inv}(\tilde{N}) = [l(\lambda), r(\lambda)]$ for $\lambda \in [0, \lambda_1]$, and $\mathrm{inv}(\tilde{N}) = \emptyset$ for $\lambda > \lambda_1$. Now for $\lambda \in [0, \lambda_1]$ the exit set $\tilde{n}^+$ consists of the left boundary point of $\tilde{N}$, thus $\tilde{N}/\tilde{n}^+ \sim [c, d]/\{c\} \sim \{c\}/\{c\}$ has the homotopy type called $\overline{0}$, the (pointed) one-point space. Since from the very definition also $\emptyset/\emptyset$ ($\emptyset$= empty set) has homotopy type $\overline{0}$, we find that $h(\mathrm{inv}(\tilde{N});f_\lambda)=\overline{0}$ for $\lambda\in[0,\lambda_2]$.

This illustrates how the Conley index is defined, and of course the definition is not restricted to one-dimensional systems. The usefulness of this index comes from its properties.

(a) (Existence Principle.) If $I=\mathrm{inv}(U)$ and $h(I)\neq\overline{0}$, then $\mathrm{inv}(U)\neq\emptyset$. In words: there exists an initial value in the isolating neighborhood U such that the whole forward and backward trajectory through this initial value is contained in the isolating neighborhood.

(b) (Continuation Theorem.) In a parameter-dependent problem $\dot{x}=f_\lambda(x)$, if the generated flow is continuous in $\lambda\in[a,b]$, and if U is an isolating neighborhood on all the λ-levels between a and b, then the Conley index of the generated isolated invariant set is independent of λ, i.e., $h(\mathrm{inv}(U);f_\lambda)$ is constant.

Both sets N and $\tilde{N}$ from above may serve as an example of (b), since $h(\mathrm{inv}(N);f_\lambda)=h(\{a(\lambda)\};f_\lambda)=\Sigma^0$ for $\lambda\in[0,\lambda_2]$ and $h(\mathrm{inv}(\tilde{N});f_\lambda)=\overline{0}$ for $\lambda\in[0,\lambda_2]$. As an application of (b), we get a criterion for bifurcation. Suppose that zero is a fixed-point of all the systems $\dot{x}=f_\lambda(x)$ for $\lambda\in[a,b]$ (i.e., $f_\lambda(0)=0$) such that $\{0\}$ is an isolated invariant set. Then we have

(c) (A Criterion for Bifurcation.) In the above set-up, if $h(\{0\};f_a)\neq h(\{0\};f_b)$, then some $\lambda_0\in[a,b]$ is a bifurcation point for small non-trivial solutions of $\dot{x}=f_\lambda(x)$. This means that for $\varepsilon>0$ there exists $\lambda\in[a,b]$ and a global non-trivial solution x of $\dot{x}=f_\lambda(x)$ such that $|\lambda-\lambda_0|+|x(t)|\leq\varepsilon$ for all times $t\in\mathbf{R}$.

Therefore if λ varies from a to b, a jump of the Conley index causes bifurcation. To prove this, it follows directly from (b) and the definition of a bifurcation point that in case there would be no bifurcation point the function $\lambda\mapsto h(\{0\};f_\lambda)$ were (locally) constant on $[a,b]$, a contradiction. The Conley index also enjoys an addition property, i.e., the index of a disjoint union of two isolated invariant sets is the "sum" of the respective homotopy types.

Now we turn to a generalization of this powerful apperatus to non-smooth systems. For illustration we deal with the example

$$\ddot{x}+\lambda\dot{x}+x^3+\varphi(x,\dot{x})\,\mathrm{sgn}(\dot{x})=0, \tag{16}$$

with some function $\varphi(x,\dot{x})$ and the bifurcation parameter $\lambda\in\mathbf{R}$. Recently we learned that equations of type (16) are used to model wings of insects; see [H. Scharstein (1998a), H. Scharstein (1998b)]. Here, as in the last section, $\mathrm{sgn}(0)$ should be considered to be the whole interval $[-1,1]$. Thus if we rewrite (16) with $x'=v$ as the first-order system

$$(x,v)'=F_\lambda(x,v)=\big(v,-\lambda v-x^3-\varphi(x,v)\,\mathrm{sgn}(v)\big), \tag{17}$$

then the right-hand side $F_\lambda(x,v)$ takes as values an interval for $v=0$. Indeed, we have

$$F_\lambda(x,0)=\{0\}\times[-\varphi(x,0)-x^3,\varphi(x,0)-x^3]. \tag{18}$$

Since this is a set, we will more exactly write $\dot{y}\in F_\lambda(y)$, but note this is just (17), with $y=(x,v)$, for $v\neq0$. These "multivalued differential equations" $\dot{y}\in F_\lambda(y)$ (also called "differential inclusions") can have several solutions; whence we first need to adapt the definition of an isolated invariant set. We introduce

$$\begin{aligned}\mathrm{inv}(U;F_\lambda) := \{y_0 : &\text{there exists } \textit{some} \text{ solution of } \dot{y}\in F_\lambda(y), y(0)=y_0,\\ &\text{that remains in } U \text{ for all times}\}.\end{aligned}$$

Therefore it is only required that one of all possible solutions stayed in U. Then I is called an isolated invariant set, if $I=\mathrm{inv}(U)\subset U$ for some isolating neighborhood U, and we want to define a (generalized) Conley index $H(I;F)$ that simplifies to the Conlex index $h(I;f)$ introduced before if restricted to usual dynamical systems $\dot{y}=f(y)$ instead of the multi-valued $\dot{y}\in F(y)$.

The key to such extension of Conlex index theory is provided by the continuation theorem, cf. (b) above, that may also be interpreted as follows: if a smooth right-hand side can be deformed continuously to another smooth right-hand side, then the Conley index does not change along the deformation. Now it is known that set-valued maps like $F_\lambda(x,v)$ in (17) can be well-approximated through smooth ones; think just of e.g. approximating

$$\mathrm{sgn}(v)=\begin{cases} -1 & : \quad v<0\\ [-1,1] & : \quad v=0\\ 1 & : \quad v>0\end{cases} \tag{19}$$

with, say, $(2/\pi)\arctan(v/\varepsilon)$, $\varepsilon\to 0^+$. Hence given an error bound ε we will find, roughly speaking, a usual smooth function $f:\mathbf{R}^n\to\mathbf{R}^n$ such that

$$f(y)\in F(y)+\text{“small error of order } \varepsilon\text{”}, \tag{20}$$

and for this f we already do have a Conley index $h(I;f)$, by the classical theory. Therefore it is tempting to define

$$H(I;F):=h(I;f). \tag{21}$$

Now (21) is indeed meaningful, since the continuation theorem can be used to prove that this assignment is independent of the choice of the approximation f, if the error bound ε is sufficiently small. In fact, for two approximants f_0 and f_1 it turns out that the deformation $\lambda\mapsto f_\lambda=(1-\lambda)f_0+\lambda f_1$ of f_0 into f_1 is appropriate for an application of the continuation theorem, and thus $h(I;f_0)=h(I;f_1)$. Hence $H(I;F)$ is well-defined. It also can be shown that an isolated invariant set I for F is an isolated invariant set for the approximation f; see [M. Kunze et al. (1998)] or [M. Kunze (1998b), Sec. 6] for the technicalities.

To summarize, we now have defined a Conley index $H(I;F)$ for problems of the form $\dot{y}\in F(y)$, and it can be seen that all the good properties of the classical Conley index h are transferred to H via (21). Hence we have

Lemma 2 *The existence principle (a), the continuation theorem (b), and the criterion for bifurcation (c) also hold for the generalized Conley index H.*

As an illustration, we are going to show how the criterion for bifurcation (c) can be used in (16) with some continuous $\varphi(x,v)$ being superlinear in v, i.e., we require $|\varphi(x,v)|\le C|v|^{1+\alpha}$ for some constants $\alpha>0$ and $C>0$. Written in the form (17), we first observe that $(x,v)=(0,0)$ is an equilibrium for all $\lambda\in\mathbf{R}$, since $\varphi(0,0)=0$, and therefore $F_\lambda(0,0)=(0,0)$, cf. (18). Hence we have to verify that $\{(0,0)\}$ is an isolated invariant set for F_λ and that $H(\{(0,0)\},F_\lambda)$ jumps when λ crosses zero.

We first consider the case $\lambda>0$. Define the Lyapunov function $V(x,v)=\frac{1}{2}v^2+\frac{1}{4}x^4$, and $U_r=\{(x,v)\in\mathbf{R}^2 : V(x,v)<r\}$. Then we have the bounds

$$2C|v|^\alpha\le\lambda \quad\text{and}\quad |\varphi(x,v)|\le C|v|^{1+\alpha}, \quad (x,v)\in U_r, \tag{22}$$

if r (depending on λ) is chosen small enough. We claim that $\mathrm{inv}(U_r;F_\lambda)=\{(0,0)\}$, i.e., $\{(0,0)\}$ is an isolated invariant set for F_λ with isolating neighborhood U_r. To see this, assume that there is a solution $X=(x,\dot{x}):\mathbf{R}\to U_r$ of (17) with $X(0)=(x_0,v_0)\neq(0,0)$. Then we can write (16) in the form

$$\ddot{x}+\lambda\dot{x}+x^3+\varphi(x,\dot{x})\,w(t)=0,$$

with some function $w:\mathbf{R}\to[-1,1]$ that realizes $\mathrm{sgn}(\dot{x}(t))$, i.e., $w(t)\in\mathrm{sgn}(\dot{x}(t))$, the multi-valued signum function from (19). By definition of $w(t)$ and (22) we obtain

$$\begin{aligned}\frac{d}{dt}V(X(t)) &= \dot{x}\,[\ddot{x}+x^3]=\dot{x}\,[-\lambda\dot{x}-\varphi(x,\dot{x})\,w(t)]\\ &= -\lambda\,|\dot{x}|^2-\varphi(x,\dot{x})\,|\dot{x}|\le-\lambda\,|\dot{x}|^2+C\,|\dot{x}|^{2+\alpha}\\ &\le-(\lambda/2)\,|\dot{x}|^2.\end{aligned}$$

It may be shown that this enforces the solution to be identically zero; in particular $(x_0,v_0)=(0,0)$, a contradiction. Hence $\{(0,0)\}$ is an isolated invariant set for F_λ when $\lambda>0$. On the other hand, in case that $\lambda<0$ a similar argument may be used, since then with analogous notation

$$\frac{d}{dt}V(X(t))=-\lambda\,|\dot{x}|^2-\varphi(x,\dot{x})\,|\dot{x}|\ge-\lambda\,|\dot{x}|^2-C\,|\dot{x}|^{2+\alpha}\ge-(\lambda/2)\,|\dot{x}|^2.$$

Next we will calculate $H(\{(0,0)\};F_\lambda)=\Sigma^0$ in the case $\lambda>0$. For that, we have to use the definition of H in (21) and approximate the multi-valued map. Let $w_\varepsilon(v)=(2/\pi)\arctan(v/\varepsilon)$ be the standard approximations of $\mathrm{sgn}(v)$, see (19) above; whence

$$f_\varepsilon(x,v)=\Big(v,\,-\lambda v-x^3-\varphi(x,v)\,w_\varepsilon(v)\Big)$$

serves as an approximation of F_λ in the (somewhat rough) sense of (20). Thus

$$H(\{(0,0)\};F_\lambda)=h(\mathrm{inv}(U_r);f_\varepsilon)$$

for $\varepsilon>0$ small by definition (21). Since $|w_\varepsilon(v)|\le 1$, $V(x,v)$ also can be used as Lyapunov function for the system $(x,v)'=f_\varepsilon(x,v)$. As above we obtain $\mathrm{inv}(U_r;f_\varepsilon)=\{(0,0)\}$. Therefore $B=U_r$ is a corresponding isolating block with immediate exit set $b^+=\emptyset$. Thus

$$H(\{(0,0)\};F_\lambda)=h(\mathrm{inv}(U_r);f_\varepsilon)=[B/b^+]=\Sigma^0,$$

because $B=U_r$ can be contracted to a single point.

In case that $\lambda<0$, we may proceed similarly. Then $b^+=\partial B$ is the whole boundary of B, and identifying this to a point we obtain the (pointed) two-sphere in $\mathbf{R}^3$ denoted by Σ^2. Consequently we have shown

$$H(\{(0,0)\};F_\lambda)=\Sigma^0\;(\lambda>0) \quad\text{and}\quad H(\{(0,0)\};F_\lambda)=\Sigma^2\;(\lambda<0).$$

This jump of the index implies by the bifurcation criterion (c)

Theorem 2 *If $|\varphi(x,v)| \leq C|v|^{1+\alpha}$, then $\lambda_0=0$ is a bifurcation point for small non-trivial solutions of (16).*

Of course this theorem also hold with x^3 replaced by x^{2k+1} in (16), for some integer k. Numerically, it is seen that (16) has periodic solutions for $\lambda<0$ close to zero; cf. [M. Kunze et al. (1998)]. More sophisticated topological arguments, mainly resting on [J.R. Ward (1996), J.R. Ward (1998)], even allow to prove a global bifurcation theorem for problems like (16). This means the following: the bifurcation point λ_0 with corresponding solution through the initial value $x_0=0\in\mathbf{R}^n$ lies in some connected component $\mathcal{C}$ of the space $[a,b]\times\mathbf{R}^n$, where in "horizontal" direction the parameter space $\lambda\in[a,b]$ is drawn, and in "vertical" direction the initial values to solutions. Then the component $\mathcal{C}$ has two possibilities, namely either it is unbounded in $[a,b]\times\mathbf{R}^n$, or it has to meet the boundary $\{a,b\}\times\mathbf{R}^n$.

1.3 Numerical analysis of a friction oscillator

Both preceding sections relied on mathematically rigorous methods for non-smooth dynamical systems. However, in complicated problems from e.g. technical engineering these methods will only be applicable to much simplified models, and one rather has to stick to a numerical analysis of the system at hand to obtain results about the full system. It is also to be expected that new bifurcation phenomena occur which are not known for smooth systems, mainly since trajectories can "graze" at some discontinuity surface; cf. Fig. 1 above.

We are going to consider a specific example of this, a single-degree-of-freedom friction oscillator; see [M. Kunze and T. Küpper (1997)] or [M. Kunze (1998b), Sec. 14]. The system consists of a mass on a rough horizontal belt which turns with constant velocity $v_0=1$. The mass is attached to a spring and periodically excited by a forcing $p(t)=\gamma\cos(\eta t)$. Depending on the forcing, the mass can thus stick to the belt (if it has velocity $\dot{x}=1$), or it can move faster or less fast than the belt. The governing equation for this system is, after a suitable normalization (also ensuring that $(x,\dot{x})=(0,0)$ becomes a stationary solution of the unforced system with $\gamma=0$),

$$\ddot{x}+x+a[\mu(1)+\mu(\dot{x}-1)\,\mathrm{sgn}(\dot{x}-1)]=\gamma\cos(\eta t) \tag{23}$$

for $t\in\mathbf{R}$ with a parameter $a>0$; cf. also [K. Popp and P. Stelter (1990a), K. Popp and P. Stelter (1990b), P. Stelter (1990)]. The friction characteristic $\mu(v)$ is taken as

$$\mu(v)=\frac{\mu_0-\mu_1}{1+\lambda_0\,|v|}+\mu_1+\lambda_1\,|v|^2 \tag{24}$$

for $v\in\mathbf{R}$, so that in particular $\mu(-v)=\mu(v)$. We shall also assume that $\mu_0>\mu_1>0$, i.e., the static friction coefficient exceeds the kinetic friction coefficient. In (23) resp. (24),

$$a=10,\quad \mu_0=0.4,\quad \mu_1=0.1,\quad \lambda_0=1.42\quad\text{and}\quad \lambda_1=0.01$$

always will be fixed (see [K. Popp and P. Stelter (1990b)]), whereas amplitude γ and frequency $(2\pi)/\eta$ of the forcing are allowed to vary; they will be used as bifurcation parameters. In (23), $\mathrm{sgn}(v)$ again has to be considered as the multi-valued function from (19) for $v=0$.

After having described the setup for the problem we now study bifurcations w.r.t. $\gamma\geq 0$ and $\eta\geq 0$. An overview of the different possible behaviours of the friction oscillator in different regions of (γ,η) can be found in [M. Kunze and T. Küpper (1997), Fig. 6],

Figure 4: Construction of P and h

Figure 5: Bifurcation diagram for $\gamma = 0.7$ and $\eta \in [1, 1.5]$

[M. Kunze (1998b), Fig. 13], or [A. Wiederhöft (1994)]. Here we will restrict ourselves to highlighting a somewhat strange new phenomenon numerically found in [M. Kunze and T. Küpper (1997)], a "period doubling-undoubling-doubling...etc."-sequence. Accordingly, in (23) we only deal with parameters

$$\gamma = 0.7, \quad \eta \in [1, 1.5].$$

In order to obtain qualitative information on the whole system (23), it will be convenient to reduce to a suitable one-dimensional Poincaré return-map constructed as indicated in Fig. 4. To explain Fig. 4, note first that similarly to (5) the mass can be forced to stick to the belt during some time interval $[t_*, t^*]$. Then (23) gives $b^-(t) \leq x(t) \leq b^+(t)$ in $[t_*, t^*]$, with

$$b^\pm(t) = -a\mu(1) \pm a\mu_0 + \gamma\cos(\eta t),$$

and the solution moves on a straight line in the sliding motion region

$$\mathcal{S} = \{(x, 1, t) : b^-(t) \leq x \leq b^+(t)\}$$

until it cannot stay in $\mathcal{S}$ any longer. Note that the second component equals one in $\mathcal{S}$ since the mass has the same velocity as the belt. To define the return map, let

$$\mathcal{R} = \{(b^+(t), 1, t) : t \in \mathbf{R}\} \subset \mathcal{S}$$

denote the "right-hand side" of $\mathcal{S}$. It is found numerically that solutions starting in $\mathcal{R}$ at time $t = t_0$ will return to $\mathcal{R}$ at some later time $t_1 > t_0$. Since solutions are unique to the right, the Poincaré map $P : \mathcal{R} \to \mathcal{R}$,

$$(b^+(t_0), 1, t_0) \mapsto (b^+(t_1), 1, t_1) \mapsto \text{next exit point from } \mathcal{R} \text{ after } \mathrm{t} = \mathrm{t}_1$$

is therefore well-defined; see Fig. 4. Note that due to the geometry of $\mathcal{S}$, at some boundary points it may be impossible to exit from $\mathcal{S}$, so that the next possible exit point $(x_2, 1, t_2)$ may in fact have $t_2 > t_1$. We also remark that the construction of P is analogous to the construction of the "impact map" in impacting systems; cf. [C.J. Budd and F. Dux (1994), G.S. Whiston (1987), C.J. Budd and A.G. Lee (1995)]. Because the forcing $\gamma\cos(\eta t)$ has period $(2\pi)/\eta$, the map P naturally carries over to a one-dimensional $h : [0, (2\pi)/\eta] \to [0, (2\pi)/\eta]$, as is explained in Fig. 4. Of course, in general h depends on γ and η, and it will be thought of as a map from $[0, 2\pi]$ to itself by identifying h with the rescaled $\tilde{h} : [0, 2\pi] \to [0, 2\pi]$, $\tau \mapsto \eta h(\tau/\eta)$.

The bifurcation diagram Fig. 5 then shows the attractor of this one-dimensional rescaled maps iterated, in dependence on $\eta = [1, 1.5]$.
E.g. for $\eta = 1.025$ we observe an attracting fixed point, for $\eta = 1.15$ we find a two-periodic solution, and the more points we do have on a vertical the more complicated is the dynamics

Figure 6: Bifurcation diagram for $\gamma = 0.7$ and $\eta \in [1.086, 1.094]$

Figure 7: Bifurcation diagram for $\gamma = 0.7$ and $\eta \in [1.15, 1.25]$

of $h = h_\eta$. The structurally stable regions of the periodic solutions are quite large, and also period doublings leading to "chaos" are possible. This is illustrated with Fig. 6, an enlargement of the region $\eta \in [1.086, 1.094]$ from Fig. 5.

Next we have a closer look on the parameter region $\eta \in [1.15, 1.25]$ from Fig. 5, which is enlarged in Fig. 7 below. It can be seen that, starting from $\eta_* \approx 1.175$, consecutive period-doublings and period-undoublings from a 2-periodic orbit to a 4-periodic orbit and back are established. To our knowledge, this kind of bifurcation has not been observed before, and we remark that it results from "grazing", i.e., from a tangential arrival of trajectories at the discontinuity surface.

To explain this phenomenon in greater detail, the second resp. fourth iterate of $h_\eta : [0, 2\pi] \to [0, 2\pi]$ for $\eta = \eta_*$ and $\eta = 1.19$ are given in Fig. 8 below, where we have $\gamma = 0.7$.

The corresponding phase portraits are displayed in Fig. 9 and Fig. 10.

The fixed point at $\tau_* \approx 1.681$ of $h^2_{\eta_*}$ (see Fig. 8) corresponds to the 2-periodic solution shown in the second phase portrait. Since the trajectories arrive tangentially at the discontinuity surface, obviously the system cannot be structurally stable for $\gamma = 0.7$ and $\eta = \eta_*$.

It turns out that for $\eta \neq \eta_*$ to the right of and near to η_*, the corresponding fixed point τ near τ_* of h^2_η has lost its stability (invisible in the graph of h^2_η), but τ has become stable for h^4_η, so that a stable 4-periodic orbit appears; cf. the phase portrait. This is possible since the slope of $h^4_{\eta_*}$ at $\tau = \tau_*$ equals -1. Approaching the next critical $\eta \approx 1.198$ from the left, this process is reversed, and so on.

We note that grazing bifurcations also have been studied before, e.g. in impacting oscillator systems; see [A. Nordmark (1991), C.J. Budd (1996), W. Chin et al. (1994)], and the references therein. However, a bifurcation structure like in Fig. 7 as a consequence of grazing seems to be new.

To conclude this small numerical section, in Fig. 11 below we give the Lyapunov exponents

h^2_η for $\eta = \eta_*$

h^2_η for $\eta = 1.19$

h^4_η for $\eta = \eta_*$

h^4_η for $\eta = 1.19$

Figure 8: h^2_η and h^4_η for $\eta = \eta_*$ and $\eta = 1.19$, with $\gamma = 0.7$

Figure 9: Phase portrait for $\eta = \eta_*$ and $\gamma = 0.7$

Figure 10: Phase portrait for $\eta = 1.19$ and $\gamma = 0.7$

corresponding to Fig. 7. They also indicate a change of stability at $\eta_* \approx 1.175$.

References

[M.A. Aizerman and F.R. Gantmakher (1958)] Aizerman, M.A. and Gantmakher, F.R. (1958) "On the stability of periodic motions", *J. Appl. Math. Mech.* **22**, 1065.

[J.P. Aubin and A. Cellina (1984)] Aubin, J.P. and Cellina, A. (1984) *Differential Inclusions*, Springer, Berlin-Heidelberg-New York.

[S.F. Bockman (1991)] Bockman, S.F. (1991) "Lyapunov exponents for systems described by differential equations with discontinuous right-hand sides", in *Proc. of the American Control Conference*, Vol. 2, 1673.

[C.J. Budd (1996)] Budd, C.J. (1996) "Non-smooth dynamical systems and the grazing bifurcation", in *Nonlinear Mathematics and its Applications, Guildford 1995*, Cambridge University Press, Cambridge, 219.

[C.J. Budd and F. Dux (1994)] Budd, C.J. and Dux, F. (1994) "Intermittency in impact oscillators close to resonance", *Nonlinearity* **7**, 1191.

[C.J. Budd and A.G. Lee (1995)] Budd, C.J. and Lee, A.G. (1995) "Double impact orbits of a single-degree-of-freedom impact oscillators subject to periodic forcing of odd frequency", preprint.

[W. Chin et al. (1994)] Chin, W., Ott, E., Nusse, H.E. and Grebogi, C. (1994) "Grazing bifurcations in impact oscillators", *Phys. Rev. E* **50**, 4427.

[C. Conley (1978)] Conley, C. (1978) *Isolated Invariant Sets and the Morse Index*, CBMS **38**, American Math. Soc., Providence/RI.

[K. Deimling (1992)] Deimling, K. (1992) *Multivalued Differential Equations*, de Gruyter, Berlin-New York.

[K. Deimling (1994)] Deimling, K. (1994) "Resonance and Coulomb friction", *Differential Integral Equations* **7**, 759.

[K. Deimling and P. Szilágyi (1994)] Deimling, K. and Szilágyi, P. (1994) "Periodic solutions of dry friction problems", *Z. Angew. Math. Phys.* **45**, 53.

Figure 11: Lyapunov exponents for $\gamma = 0.7$ and $\eta \in [1.15, 1.25]$

[M. Kunze (1998a)] Kunze, M. (1998a) "Unbounded solutions in non-smooth dynamical systems at resonance", *Z. Angew. Math. Mech.* **78**, Supplement 3, S985.

[M. Kunze (1998b)] Kunze, M. (1998b) *Non-Smooth Dynamical Systems*, Habilitationsschrift, Universität Köln.

[M. Kunze and T. Küpper (1997)] Kunze, M. and Küpper, T. (1997) "Qualitative analysis of a non-smooth friction-oscillator model", *Z. Angew. Math. Phys.* **48**, 1.

[M. Kunze et al. (1998)] Kunze, M., Küpper, T. and Li, J. (1998) "On the application of Conley index theory to non-smooth dynamical systems", submitted to *Differential Integral Equations.*

[M. Kunze and B. Michaeli (1995)] Kunze, M. and Michaeli, B. (1995) "On the rigorous applicability of Oseledets' theorem to obtain Lyapunov exponents for non-smooth dynamical systems", to appear in *Proc. 2. Int. Conf. Diff. Equ., Marrakesh 1995*, Eds. Arino, O. and Vanderbauwhede, A.

[W. de Melo and S. van Strien (1993)] de Melo, W. and van Strien, S. (1993) *One-Dimensional Dynamics*, Springer, Berlin-Heidelberg-New York.

[B. Michaeli (1998)] Michaeli, B. (1998) *Lyapunov-Exponenten bei nichtglatten dynamischen Systemen* (in German), Ph.D. thesis, Universität Köln.

[P.C. Müller (1995)] Müller, P.C. (1995) "Calculation of Lyapunov exponents for dynamic systems with discontinuities", *Chaos, Solitons & Fractals* **5**, 1671.

[A. Nordmark (1991)] Nordmark, A. (1991) "Non-periodic motion caused by grazing incidence in an impact oscillator", *J. Sound Vibration* **145**, 279.

[V.I. Oseledets (1968)] Oseledets, V.I. (1968) "A multiplicative ergodic theorem. Ljapunov characteristic numbers for dynamical systems", *Trans. Moscow Math. Soc.* **19**, 197.

[K. Popp and P. Stelter (1990a)] Popp, K. and Stelter, P. (1990a) "Nonlinear oscillations of structures induced by dry friction", in *Nonlinear Dynamics in Engineering Systems, IUTAM Symposium Stuttgart 1989*, Ed. Schiehlen, W., Springer, Berlin-Heidelberg-New York, 233.

[K. Popp and P. Stelter (1990b)] Popp, K. and Stelter, P. (1990b) "Stick-slip vibrations and chaos", *Phil. Trans. Roy. Soc. London A* **332**, 89.

[H. Scharstein (1998a)] Scharstein, H. (1998a) "Der optimale Arbeitsbereich von Schlagflügelantrieben" (in German), preprint.

[H. Scharstein (1998b)] Scharstein, H. (1998b) "Kräfte- und Leistungsbilanz bei der künstlichen Schlagbewegung einzelner Insektenflügel" (in German), in *Technische Biologie und Bionik 4, München 1998*, Eds. Nachtigall, W. and Wisser, A., Gustaf Fischer Verlag, Stuttgart.

[J. Smoller (1994)] Smoller, J. (1994) *Shock Waves and Reaction-Diffusion Equations*, 2nd edition, Springer, Berlin-Heidelberg-New York.

[P. Stelter (1990)] Stelter, P. (1990) *Nichtlineare Schwingung reibungserregter Strukturen* (in German), Ph.D. thesis, Universität Hannover.

[J.R. Ward (1996)] Ward, J.R., Jr. (1996) “Bifurcating continua in infinite dimensional dynamical systems and applications to differential equations”, *J. Differential Equations* **125**, 117.

[J.R. Ward (1998)] Ward, J.R., Jr. (1998) “Global bifurcation of periodic solutions to ordinary differential equations”, *J. Differential Equations* **142**, 1.

[G.S. Whiston (1987)] Whiston, G.S. (1987) “The vibro-impact response of a harmonically excited and preloaded one-dimensional linear oscillator”, *J. Sound Vibration* **115**, 303.

[A. Wiederhöft (1994)] Wiederhöft, A. (1994) *Der periodisch erregte Einmassenreibschwinger* (in German), Diploma Thesis, Universität Köln.

[L.-S. Young (1998)] Young, L.-S. (1998) “Developments in chaotic dynamics”, *Notices of the American Math. Soc.* **45** (10), 1318.