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Periodic Solutions of Conservative Non-Smooth Dynamical Systems

We consider conservative non-smooth dynamical systems of the type $\ddot{x}+x+a\,\mathrm{sgn}(x)=p(t)$ with an ω-periodic forcing p, and we first prove the existence of periodic solutions in the non-resonant case for the corresponding differential inclusion. Moreover, in the resonant case $\omega=2\pi$, we give necessary and sufficient conditions on a and the first Fourier coefficient of p for periodic solutions to exist.

1. Introduction

There are many physically relevant dynamical systems, in particular in mechanics, with non-smooth nonlinearities. Typically, such discontinuities arise in the context of modelling effects like impacts of Coulomb friction, cf. [7, 11]. Roughly speaking, two different types of second-order systems can be distinguished, namely dissipative and conservative ones, depending on the phase portrait in $(x,\dot{x})$-plane of the unforced system. Dissipative non-smooth systems will have some kind of global attractor (e.g., a whole interval of fixed points, cf. [9; Fig. 1]), whereas solutions of unforced conservative systems are expected to form closed periodic curves, which, however, may be non-smooth, cf. [10; Fig. 1].

In this note, we will consider the problem to prove existence of periodic solutions of the conservative system

$$\ddot{x}+x+a\,\mathrm{sgn}(x)=p(t)\quad \text{a.e.,} \tag{1}$$

where p is a continuous and ω-periodic forcing, and $a>0$ is a parameter giving the size of the jump-discontinuity of the restoring force $f_a(x)=x+a\,\mathrm{sgn}(x)$. A Hamiltonian for (1) with $p=0$ is given through

$$H_a(x,\dot{x})=\frac{1}{2}\,\dot{x}^2+\frac{1}{2}\Big((x_+ +a)^2+(x_- +a)^2-a^2\Big),$$

where $x_+=\max\{x,0\}$ and $x_-=\max\{-x,0\}$. In fact we will treat a weakend form of (1), namely the differential inclusion

$$\ddot{x}+x+a\,\mathrm{Sgn}(x)\ni p(t)\quad \text{a.e.,} \tag{2}$$

which has the advantage of not fixing the value of sgn at $x=0$, but allowing all values in $[-1,1]$, i.e., for every $x\in\mathbb{R}$, $\mathrm{Sgn}(x)\subset\mathbb{R}$ is a set given through

$$\mathrm{Sgn}(x)=\begin{cases}\{-1\} & : & x<0\\ [-1,1] & : & x=0\\ \{1\} & : & x>0\end{cases}\ .$$

As in the case of smooth conservative systems, concerning the existence of periodic solutions of (2) there is a big difference between the non-resonant case $\omega\neq 2\pi k$ for all $k\in\mathbb{N}$ and the opposite resonant case. It is more or less no problem to find periodic solutions in the non-resonant case (cf. Theorem 1 below), but it is know from [8; Example 2] that for $\omega=2\pi$ all solutions of (2) will be unbounded in the $(x,\dot{x})$-plane, provided $\left|\int_0^{2\pi}p(t)e^{it}\,dt\right|>4a$, meaning that the gap-size a is small compared to the first Fourier coefficient of p. Thus in particular there will be no periodic solutions in this case, and a necessary condition is

$$\left|\int_0^{2\pi}p(t)e^{it}\,dt\right|\leq 4a. \tag{3}$$

This situation will be analyzed in Theorem 2 below, and it will be shown that $|\ldots|<4a$ in (3) is sufficient to obtain periodic solutions, whereas in case that $|\ldots|=4a$ one can give an additional necessary and sufficient condition for such periodic solutions to exist.

The proofs of these results are similar to [3], where dissipative inclusions $\ddot{x}+x+\mu\,\mathrm{Sgn}(\dot{x})\ni p(t)$ a.e. have been treated, and it should also be remarked that related results can be found in the recent paper [4]. The reader may consult [1] or [2] for the necessary background on differential inclusions, cf. also [5].

2. Periodic Solutions

Theorem 1. *If $\omega \neq 2\pi k$ for all $k \in \mathbb{N}$, then (2) has a ω-periodic solution for every $a > 0$.*

Proof. The claim is obtained along the lines of [6], cf. [2; Problem 13.4] and also [3; Section 2] for a sketch, since with $y = (y_1, y_2) = (x, \dot{x})$ we may rewrite (2) as

$$y' \in Ay + F(t,y) \quad \text{a.e.,} \quad \text{with} \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad F(t,y) = \{0\} \times (p(t) - a\,\mathrm{Sgn}(y_1)),$$

and the only ω-periodic solution of $y' = Ay$ is $y = 0$.

Next we turn to the (first) resonant case.

Theorem 2. *Consider $\omega = 2\pi$. Then (3) is a necessary condition for (2) to have a 2π-periodic solution. Conversely, if " $<$ " holds in (3), then (2) has a 2π-periodic solution. Finally, if " $=$ " in (3), then a 2π-periodic solution exists if and only if*

$$\int_0^\pi \sin(\pi - s)\, p(s - \tau - \pi/2)\, ds = 2a \tag{4}$$

with $\tau \in [0, 2\pi[$ being the argument of $\displaystyle\int_0^{2\pi} p(t)\, e^{it}\, dt \in \mathbb{C}$. In this case there are infinitely many (explicitly known) such solutions.

Proof. If x is a solution of (2), then

$$\ddot{x}(t) + x(t) + a\, w(t) = p(t) \quad \text{a.e. in } J = [0, 2\pi], \quad \text{with} \quad w(t) \in \mathrm{Sgn}(x(t)) \quad \text{a.e. in } J \tag{5}$$

for some measurable w, and thus in particular $|w(t)| \leq 1$ a.e. We have already mentioned in the introduction, that (3) is a necessary condition, but this can be seen also more directly without the detour on unbounded solutions. Indeed, if x is 2π-periodic, then multiplication of the equation in (5) with $\cos t$ resp. $\sin t$ and integration over J yields $a \int_0^{2\pi} w(t)\, e^{it}\, dt = \int_0^{2\pi} p(t)\, e^{it}\, dt$. If we choose the unique $\tau \in [0, 2\pi[$ with $e^{i\tau} \left(\int_0^{2\pi} w(t)\, e^{it}\, dt \right) = \left| \int_0^{2\pi} w(t)\, e^{it}\, dt \right|$, then $\left| \int_0^{2\pi} w(t)\, e^{it}\, dt \right| = \int_0^{2\pi} w(t) \cos(t + \tau)\, dt$, and this shows the necessity of (3), since $|w(t)| \leq 1$ a.e. and $\int_0^{2\pi} |\cos(t + \tau)|\, dt = 4$.

Next we suppose that " $=$ " holds in (3) and that there exists a 2π-periodic solution x. With w and τ choosen as above it follows that $\int_0^{2\pi} w(t) \cos(t + \tau)\, dt = 4$, and hence $\int_0^{2\pi} [\mathrm{sgn} \cos(t + \tau) - w(t)] \cos(t + \tau)\, dt = 0$. Because the integrand is ≥ 0 a.e., we obtain $w(t) = \mathrm{sgn} \cos(t + \tau)$ a.e. Define $y, \psi : [0, 2\pi] \to \mathbb{R}$ by $y(t) = x(t - \tau - \pi/2)$ and $\psi(t) = p(t - \tau - \pi/2)$. Then (5) gives

$$\ddot{y}(t) + y(t) + a\, \bar{w}(t) = \psi(t) \quad \text{a.e. in } J, \quad \text{with} \quad \bar{w}(t) := \mathrm{sgn} \sin t \in \mathrm{Sgn}(y(t)) \quad \text{a.e. in } J. \tag{6}$$

The latter and the continuity of y imply $y(t) \geq 0$ in $[0, \pi]$ resp. $y(t) \leq 0$ in $[\pi, 2\pi]$. Thus $y(0) = y(\pi) = 0$ and

$$y(t) = \beta \sin t + \int_0^t \sin(t - s)\, [\psi(s) - a\bar{w}(s)]\, ds \quad \text{for} \quad t \in [0, 2\pi] \tag{7}$$

with $\beta := y'(0)$. Since $\int_0^\pi \sin(\pi - s)\, \bar{w}(s)\, ds = 2$, condition (4) is necessary.

On the other hand, assume that " $=$ " holds in (3) and that (4) is satisfied. We define ψ as above and first derive from (3) that

$$\int_0^{2\pi} \psi(t) \sin t\, dt = 4a \quad \text{and} \quad \int_0^{2\pi} \psi(t) \cos t\, dt = 0. \tag{8}$$

For that, the continuous 2π-periodic p can be developed into a Fourier series

$$c_0 + \sum_{k=1}^{\infty} [c_k \cos kt + d_k \sin kt] \tag{9}$$

being $L^2(J)$-convergent to p, hence $|p|_2^2 = 2\pi\, c_0^2 + \pi \sum_{k=1}^{\infty} [c_k^2 + d_k^2]$, where

$$c_0 = \frac{1}{2\pi} \int_0^{2\pi} p(t)\, dt \quad \text{and} \quad \begin{pmatrix} c_k \\ d_k \end{pmatrix} = \frac{1}{\pi} \int_0^{2\pi} p(t) \begin{pmatrix} \cos kt \\ \sin kt \end{pmatrix} dt\,, \quad k \geq 1\,. \tag{10}$$

This yields $\left| \int_0^{2\pi} p(t)\, e^{it}\, dt \right| = \pi \sqrt{c_1^2 + d_1^2}$ and $\int_0^{2\pi} p(t)\, \cos(t+\tau)\, dt = \pi\, [c_1 \cos(\tau) - d_1 \sin(\tau)]$, and therefore $4a = \pi \sqrt{c_1^2 + d_1^2} = \pi\, [c_1 \cos\tau - d_1 \sin\tau]$ from (3). In particular, $(c_1, d_1) \in \mathbb{R}^2$ and $(\cos\tau, -\sin\tau) \in \mathbb{R}^2$ are linearly dependent, hence $c_1 \sin\tau + d_1 \cos\tau = 0$. Consequently, (8) follows from

$$\int_0^{2\pi} \psi(t)\, \sin t\, dt = \pi\, [c_1 \cos\tau - d_1 \sin\tau] \quad \text{and} \quad \int_0^{2\pi} \psi(t)\, \cos t\, dt = -\pi\, [c_1 \sin\tau + d_1 \cos\tau]\,.$$

Now let $\theta(t) = \int_0^t \sin(t-s)\, \psi(s)\, ds$, choose $\beta \in \mathbb{R}$ such that

$$\beta \geq \max_{t \in [0,\pi]} \left(\frac{a(1-\cos t) - \theta(t)}{\sin t} \right) \quad \text{and} \quad \beta \geq \max_{t \in [\pi, 2\pi]} \left(\frac{-a(1+3\cos t) - \theta(t)}{\sin t} \right), \tag{11}$$

and define y by (7), with $\bar{w}(t) := \operatorname{sgn} \sin t$. Note that both maxima exist in (11), due to (4) and (8). We find

$$y(t) = \begin{cases} \beta \sin t + \theta(t) - a(1-\cos t) & : \quad t \in [0,\pi] \\ \beta \sin t + \theta(t) + a(1+3\cos t) & : \quad t \in [\pi, 2\pi] \end{cases}\,,$$

and y is C^1 and 2π-periodic by (8). Moreover, by (11), $y(t) \geq 0$ in $[0,\pi]$ and $y(t) \leq 0$ in $[\pi, 2\pi]$. Hence we clearly obtain $\ddot{y} + y + a\, \mathrm{Sgn}(y) \ni \psi(t)$ a.e., and transformation back by means of $x(t) = y(t + \tau + \pi/2)$ yields a 2π-periodic solution $x = x_\beta$ of (2) for every β satisfying (11).

Finally we have to show the existence of a 2π-periodic solution of (2) in the most interesting case that $\pi \sqrt{c_1^2 + d_1^2} = \left| \int_0^{2\pi} p(t)\, e^{it}\, dt \right| < 4a$, cf. (9) and (10). For that, we consider the approximate problems

$$\ddot{x}_\varepsilon + (1+\varepsilon)\, x_\varepsilon + a\, \mathrm{Sgn}(x_\varepsilon) \ni p(t) \quad \text{a.e.}\,, \tag{12}$$

which have 2π-periodic solutions $x_\varepsilon \in W^{2,2}(J) \subset C^1(J)$ by the theorem mentioned at the beginning of the proof in (a), since every nontrivial solution of the corresponding linear homogeneous equation $\ddot{x}_\varepsilon + (1+\varepsilon)\, x_\varepsilon = 0$ has period $2\pi/\sqrt{1+\varepsilon}$. We first want to derive a uniform $L^2(J)$-bound of the solutions $\mathcal{X} := \{x_\varepsilon : \varepsilon \in (0,1]\} \subset L^2(J)$, and for that it is sufficient to show that $\mathcal{X} \subset L^2(J)$ is weakly bounded. It will be convenient to expand every x_ε into a Fourier series, like in (9), (10), with coefficients $a_{k,\varepsilon}$ and $b_{k,\varepsilon}$. These series for x_ε resp. $\dot{x}_\varepsilon$ are uniformly resp. $L^2(J)$-convergent, and we thus have

$$|x_\varepsilon|_2^2 = 2\pi\, a_{0,\varepsilon}^2 + \pi \sum_{k=1}^{\infty} [a_{k,\varepsilon}^2 + b_{k,\varepsilon}^2] \quad \text{as well as} \quad |\dot{x}_\varepsilon|_2^2 = \pi \sum_{k=1}^{\infty} k^2\, [a_{k,\varepsilon}^2 + b_{k,\varepsilon}^2]\,. \tag{13}$$

We define a continuous linear projection $P : L^2(J) \to L^2(J)$ through

$$(Py)(t) := \left(\frac{1}{\pi} \int_0^{2\pi} y(s)\, \cos s ds \right) \cos t + \left(\frac{1}{\pi} \int_0^{2\pi} y(s)\, \sin s ds \right) \sin t\,.$$

This P is also symmetric, i.e. $\langle Py, z \rangle = \langle y, Pz \rangle$, and thus also $\langle (Id - P)y, z \rangle = \langle y, (Id - P)z \rangle$, with $\langle y, z \rangle := \int_0^{2\pi} y(s)\, z(s)\, ds$ for $y, z \in L^2(J)$. Next we choose measurable functions w_ε with

$$\ddot{x}_\varepsilon(t) + (1+\varepsilon)\, x_\varepsilon(t) + a\, w_\varepsilon(t) = p(t) \quad \text{a.e. in } J\,, \quad \text{with} \quad w_\varepsilon(t) \in \mathrm{Sgn}(x_\varepsilon(t)) \quad \text{a.e. in } J\,, \tag{14}$$

hence in particular $|w_\varepsilon(t)| \leq 1$ a.e. Thus, if we also let $A_{0,\varepsilon} = \frac{1}{2\pi} \int_0^{2\pi} w_\varepsilon(s)\, ds$ and $A_{k,\varepsilon} = \frac{1}{\pi} \int_0^{2\pi} w_\varepsilon(s)\, \cos ks\, ds$ resp. $B_{k,\varepsilon} = \frac{1}{\pi} \int_0^{2\pi} w_\varepsilon(s)\, \sin ks\, ds$ for $k \geq 1$, then we have $\sup\{|A_{k,\varepsilon}| : k \in \mathbb{N}_0, \varepsilon \in (0,1]\} \leq 2$ and also $\sup\{|B_{k,\varepsilon}| : k \in \mathbb{N}, \varepsilon \in (0,1]\} \leq 2$. Additionally, (14) also implies $(1+\varepsilon)\, a_{0,\varepsilon} + a\, A_{0,\varepsilon} = c_0$ and $(1+\varepsilon-k^2)\, a_{k,\varepsilon} + a\, A_{k,\varepsilon} = c_k$ resp. $(1+\varepsilon-k^2)\, b_{k,\varepsilon} + a\, B_{k,\varepsilon} = d_k$ for $k \geq 1$. For $k \geq 2$ and $\varepsilon \in (0,1]$ we therefore obtain the estimate

$$|a_{k,\varepsilon}| = \left| \frac{c_k - a\, A_{k,\varepsilon}}{1+\varepsilon-k^2} \right| \leq \frac{2\,(a + |p|_\infty)}{|k^2 - 1 - \varepsilon|} \leq \frac{4}{k^2}\,(a + |p|_\infty) \quad \text{and also} \quad |b_{k,\varepsilon}| \leq \frac{4}{k^2}\,(a + |p|_\infty)\,.$$

This implies for $\varepsilon \in (0,1]$

$$|\dot{x}_\varepsilon|_2^2 - |x_\varepsilon|_2^2 = \pi \sum_{k=1}^{\infty} k^2 \,[a_{k,\varepsilon}^2 + b_{k,\varepsilon}^2] - 2\pi\, a_{0,\varepsilon}^2 - \pi \sum_{k=1}^{\infty} [a_{k,\varepsilon}^2 + b_{k,\varepsilon}^2] \le \pi \sum_{k=2}^{\infty} k^2 \,[a_{k,\varepsilon}^2 + b_{k,\varepsilon}^2] \le c\,, \tag{15}$$

and

$$|(Id-P)(x_\varepsilon)|_2^2 = \left| a_{0,\varepsilon} + \sum_{k=2}^{\infty} [a_{k,\varepsilon}\, \cos kt + b_{k,\varepsilon}\, \sin kt] \right|_2^2 = 2\pi\, a_{0,\varepsilon}^2 + \pi \sum_{k=2}^{\infty} [a_{k,\varepsilon}^2 + b_{k,\varepsilon}^2] \le c \tag{16}$$

for some $c > 0$ being independent of $\varepsilon \in (0,1]$. Next, we recall from [3; p. 763] that

$$\left\{ \int_0^{2\pi} w(s)\, e^{is}\, ds :\ w : J \to \mathbb{R} \text{ is measurable with } |w(s)| \le 1 \text{ a.e.} \right\} = \overline{B}_4(0) \subset \mathbb{C}.$$

Since now $\left| \int_0^{2\pi} p(s)\, e^{is}\, ds \right| < 4a$ by assumption, we find $\delta_0 > 0$ such that

$$\left| \int_0^{2\pi} [p(s) + h(s)]\, e^{is}\, ds \right| \le 4a \quad \text{for} \quad h \in L^2(J) \quad \text{with} \quad |h|_2 \le \delta_0\,.$$

We fix such $h \in L^2(J)$ with $|h|_2 \le \delta_0$. Then $a^{-1} \int_0^{2\pi} [p(s) + h(s)]\, e^{is}\, ds \in \overline{B}_4(0)$. Thus we find a measurable $w_h : J \to \mathbb{R}$ with $|w_h(s)| \le 1$ a.e. and $\int_0^{2\pi} [p(s)+h(s)]\, e^{is}\, ds = a \int_0^{2\pi} w_h(s)\, e^{is}\, ds$, in other words $P(p+h) = a\, P(w_h)$. By multiplication of (14) with x_ε and integration over J we hence obtain

$$\begin{aligned}
\langle h, x_\varepsilon \rangle &= |\dot{x}_\varepsilon|_2^2 - (1+\varepsilon)\, |x_\varepsilon|_2^2 + \langle P(p+h) - a\, w_\varepsilon,\, x_\varepsilon \rangle + \langle (Id-P)(p+h),\, x_\varepsilon \rangle \\
&= |\dot{x}_\varepsilon|_2^2 - (1+\varepsilon)\, |x_\varepsilon|_2^2 + a\, \langle P(w_h) - w_\varepsilon,\, x_\varepsilon \rangle + \langle p+h,\, (Id-P)(x_\varepsilon) \rangle \\
&= |\dot{x}_\varepsilon|_2^2 - (1+\varepsilon)\, |x_\varepsilon|_2^2 + a\, \langle w_h - w_\varepsilon,\, x_\varepsilon \rangle + \langle p+h-a\, w_h,\, (Id-P)(x_\varepsilon) \rangle \\
&\le |\dot{x}_\varepsilon|_2^2 - |x_\varepsilon|_2^2 + \langle p+h-a\, w_h,\, (Id-P)(x_\varepsilon) \rangle\,,
\end{aligned}$$

since Sgn is a monotone multivalued map, $w_h(s) \in \mathrm{Sgn}(0)$ a.e., thus $(w_\varepsilon(s) - w_h(s))\, x_\varepsilon(s) \ge 0$ a.e. by (14). Hence (15) and (16) imply that there exists a constant $C > 0$ and $\delta_0 > 0$ such that

$$\langle h, x_\varepsilon \rangle \le C\,(1 + |h|_2) \quad \text{for} \quad h \in L^2(J) \quad \text{with} \quad |h|_2 \le \delta_0 \quad \text{and} \quad \varepsilon \in (0,1]\,.$$

Therefore $\mathcal{X} = \{x_\varepsilon : \varepsilon \in (0,1]\} \subset L^2(J)$ is weakly bounded. This, together with (15), yields $\sup_{\varepsilon\in(0,1]} |x_\varepsilon|_2^2 + \sup_{\varepsilon\in(0,1]} |\dot{x}_\varepsilon|_2^2 < \infty$, and thus (14) shows that $\mathcal{X} \subset W^{2,2}(J)$ is uniformly bounded in norm. Because the embedding $W^{2,2}(J) \subset C^1(J)$ is compact, we may assume that $x_{\varepsilon_n} \to x$ in $C^1(J)$ for some sequence $\varepsilon_n \to 0$. Finally, standard arguments for upper semicontinuous multivalued equations, cf. [2; Lemma 5.1], together with (12) imply that this x solves (2).

3. References

1 Aubin, J.-P., Cellina, A.: Differential Inclusions, Springer, Berlin-Heidelberg 1984.
2 Deimling, K.: Multivalued Differential Equations, de Gruyter, Berlin-New York 1992.
3 Deimling, K.: Resonance and Coulomb friction; Differential Integral Equations **7** (1994), 759-765.
4 Fečkan, M.: Differential inclusions at resonance; preprint 1997.
5 Filippov, A.F.: Differential Equations with Discontinuous Right-Hand Sides, Kluwer, Dordrecht 1988.
6 Irisov, A.E., Tonkova, V.S., Tonkov, E.L.: Periodic solutions of differential inclusions (in Russian); Nonlinear Oscillations and Control Theory, Izhevsk **2** (1978), 3-14.
7 Kauderer, H.: Nichtlineare Mechanik, Springer, Berlin-Heidelberg 1958.
8 Kunze, M.: Unbounded solutions in non-smooth dynamical systems at resonance; to appear in Proc. GAMM Conference Regensburg 1997, as special issue of Z. Ang. Math. Mech.
9 Kunze, M.: On Lyapunov exponents for non-smooth dynamical systems with an application to a pendulum with dry friction; submitted.
10 Kunze, M., Küpper, T., You, J.: On the application of KAM theory to discontinuous dynamical systems; J. Differential Equations **139** (1997), 1-21.
11 Popp, K., Stelter, P.: Stick-slip vibrations and chaos; Phil. Trans. Roy. Soc. London A **332** (1990), 89-105.

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