Strongly generated Banach spaces and measures of non-compactness

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Abstract

To generalize the Hausdorff measure of noncompactness to other classes of bounded sets (like e.g. conditionally weakly compact or Asplund sets), we introduce Grothendieck classes. We deduce integral inequalities for quantities (called Grothendieck measures) related to these classes. As a by-product, we can answer a question concerning the measure of noncompactness for linear $T:X\to Y$ introduced by [14], and generalize a theorem about weak solutions of differential equations in Banach spaces.

1 Introduction

In [14], Lebow and Schechter defined

$$c(T)=\inf\{\|T|_Z\| : \mathrm{Z\ is\ a\ subspace\ of\ X\ with\ codim}\, Z<\infty\} \tag{1}$$

for $T\in L(X,Y)$, where X and Y are Banach spaces, and proved that $c(T)=0$ if and only if T is a compact operator.

This quantity was used in [26] to investigate the essential spectral radius of a perturbed semigroup, which is useful in describing its large-time behaviour. For his approach, Weis needed the integral inequality

$$c\Big(\int_\Omega U(\omega)\mu(d\omega)\Big)\le\int_\Omega c(U(\omega))\mu(d\omega), \tag{2}$$

for an operator-valued $U:\Omega\to L(X,Y)$, whose proof contained a gap, as shown by Voigt, cf. [24, Remark 1.1]. Despite this lack (and a measurability problem of $\omega\mapsto c(U(\omega))$, cf. Example 2 below), and the fact, that the main result on the essential spectral radius of a perturbed semigroup was independently proven by Voigt in [25] and Schlüchtermann in [21], the correctness of the above inequality has been unknown and was one main motivation for this paper.

It turned out that integral inequalities related to (2) may be obtained in an essentially more general setting, and also for other quantities which measure how far away a set is from having a certain property, e.g. like being conditionally weakly compact or Asplund. For this purpose, in Section 2 we introduce the so-called Grothendieck classes and corresponding Grothendieck measures. In fact, it turns out that an integral inequality like (2) holds for all such measures, provided the underlying Banach space is "strongly generated", a concept which generalizes the strongly WCG-spaces from [22].

In Section 3 we will show in particular, that (2) is true in general, provided one inserts a constant on the right-hand side (which does not affect the application to semigroups) and uses the upper integral. In addition, this gives a simple proof to a result of VOIGT (cf. [24]), saying that the strong integral of a compact operator-valued strongly measurable function is again compact. As an other application, we obtain an extension of a theorem of PAPAGEORGIOU (cf. [17]) concerning weak solutions of differential equations in Banach spaces.

Finally, in Section 4 we return to the investigation of Grothendieck-generated Banach spaces in the cases of weakly compact, Asplund or conditionally compact generators. We will give a complete characterization of these spaces in terms of operator factorization and locally convex topologies on the dual unit ball.

2 Grothendieck classes, strongly generated Banach spaces and Grothendieck measures

Throughout this paper, X will be a real Banach space with dual X^*, closed unit ball $B(X)$ and unit sphere $S(X)$. Moreover, $\mathcal{B}(X)$ will denote the set of all bounded subsets of X. If Y is another Banach space, then $L(X,Y)$ are the linear and bounded operators from X to Y. Finally, (Ω,Σ,μ) will denote a finite positive measure space, and $U:\Omega\to L(X,Y)$ will be called strongly measurable, if $U(\cdot)(x):\Omega\to Y$ is μ-measurable for every $x\in X$.

We shall start by introducing the basic object of our further studies, namely Grothendieck classes.

Definition 1 *A subset $\emptyset\neq\mathcal{G}(X)\subset\mathcal{B}(X)$ is called a Grothendieck class if it satisfies the following conditions:*

(a) If $\lambda\in\mathbb{R}$ and $G,H\in\mathcal{G}(X)$, then $\lambda G, G+H, G\cup H, \operatorname{aco}(G)\in\mathcal{G}(X)$.

(b) If $H\in\mathcal{G}(X)$ and $G\subset H$, then $G\in\mathcal{G}(X)$.

(c) Let $G\in\mathcal{B}(X)$. Then $G\in\mathcal{G}(X)$ iff there exists a sequence $(G_n)_{n\in\mathbb{N}}\subset\mathcal{G}(X)$ such that $G\subset\bigcap_{n\in\mathbb{N}}(G_n+\frac{1}{n}B(X))$.

There are several Grothendieck classes besides $\mathcal{B}(X)$ itself.

Example 1 *The following are Grothendieck classes:*

(a) the relatively compact sets $\mathcal{K}(X)$;

(b) the relatively weakly compact sets $\mathcal{W}(X)$, by the Krein-Šmulian theorem and the classical Grothendieck criterion;

(c) the Banach-Saks sets $\mathcal{BS}(X)$; recall that a bounded set $B \subset X$ is called a Banach-Saks set, if for every sequence (x_n) in B there is a subsequence (x_{n_j}) such that $(\frac{1}{m}\sum_{j=1}^{m} x_{n_j})$ converges in norm;

(d) the Asplund sets $\mathcal{A}(X)$; cf. [7, Lemma 5.3.4].

(e) the conditionally weakly compact sets $\mathcal{CW}(X)$; a bounded $B \subset X$ is called conditionally weakly compact, if every sequence in B admitts a weak Cauchy subsequence.

In some Banach spaces, Grothendieck classes are - in some sense - determined by a certain $G_0 \in \mathcal{G}(X)$. This will also play an important rôle in the sequel.

Definition 2 *Let X be a Banach space and $\mathcal{G}(X)$ a Grothendieck class. X is called strongly $\mathcal{G}(X)$-generated, if there exists a $G_0 \in \mathcal{G}(X)$ such that for all $G \in \mathcal{G}(X)$ and $\varepsilon > 0$ there is an $m = m(G,\varepsilon) \in \mathbb{N}$ with $G \subset m G_0 + \varepsilon B(X)$.*

G_0 is called a generating set and may be assumed to be absolutely convex, by definition of a Grothendieck class.

As an example, let us consider the case $\mathcal{G}(X) = \mathcal{K}(X)$.

Lemma 1 *X is strongly $\mathcal{K}(X)$-generated if and only if it is separable.*

Proof: For the if-part let $(x_n) \subset B(X)$ be dense and

$$G_0 = \overline{\mathrm{aco}}\{\frac{x_n}{n} : n \in \mathbb{N}\}.$$

Then G_0 is compact and a generating set. On the other hand, given a generating set G_0, it follows that $X = \overline{\mathrm{span}} G_0$. Hence X is separable since G_0 is separable as a relatively compact set. □

For strongly $\mathcal{G}(X)$-generated Banach spaces X with $\mathcal{G}(X) = \mathcal{W}(X), \mathcal{G}(X) = \mathcal{A}(X)$, or $\mathcal{G}(X) = \mathcal{CW}(X)$, examples and characterizations are given in Section 4. Then X will be called $SWCG$, SAG, or $SCWG$, respectively.

The next concept to be introduced is a quantity $\gamma(A)$ which measures how far from being in $\mathcal{G}(X)$ a set A is.

Definition 3 *Let $\mathcal{G}(X)$ be a Grothendieck class. Then*

$$\gamma : \mathcal{B}(X) \longrightarrow [0,\infty), \quad A \longmapsto \inf\{r > 0 : A \subset G + r B(X) \text{ for some } G \in \mathcal{G}(X)\}$$

is called the Grothendieck measure.

Sometimes it is useful to extend γ to 2^X by defining $\gamma(A) = \infty$ for unbounded $A \subset X$. Elementary properties of γ may be found in

Lemma 2 *Let $\lambda \in \mathbb{R}$ and $A, B \in \mathcal{B}(X)$. Then*

(a) $\gamma(\{0\}) = 0$ and $\gamma(B(X)) \leq 1$;

(b) $\gamma(\overline{A}) = \gamma(A)$;

(c) $\gamma(\lambda A) = |\lambda|\gamma(A)$ *and* $\gamma(A+B) \leq \gamma(A)+\gamma(B)$*;*

(d) $\gamma(A\cup B) = \max\{\gamma(A),\gamma(B)\}$ *and* $\gamma(\operatorname{aco} A) = \gamma(A)$*;*

(e) $A\in\mathcal{G}(X)$ *iff* $\gamma(A)=0$.

Proof: All assertions follow from the definition of γ and the properties of $\mathcal{G}(X)$. For (b) note that $\overline{G+r\,B(X)}\subset G+(r+\varepsilon)B(X)$ for $r,\varepsilon>0$. □

Let us consider some special cases.

Remark 1 *(a) For $\mathcal{G}(X)=\mathcal{K}(X)$, γ is known as the Hausdorff measure of noncompactness and called β, cf. standard references like [28] and [8], or [1]. It was shown in [14], that β is related to c, cf. (1), as follows:*

$$c(T)\leq\beta(T(B(X))\leq 2c(T).$$

Note that formally $c(\cdot)$ depends on X and Y. But if $Y_0\subset Y$ is a subspace and $T\in L(X,Y_0)$, then $c_{Y_0}(T)=c_Y(T)$; a subscript of c will be used only to emphasize the range space of the operator T under consideration.

(b) $\gamma=\gamma_w$ is called the measure of weak noncompactness if $\mathcal{G}(X)=\mathcal{W}(X)$. This example also shows that $\gamma(B(X))=0$ is possible in Lemma 2(a) what happens if and only if X is reflexive, by Lemma 2(e); cf. also [6].

The following theorem provides us with an integral inequality for γ in the case that X is strongly $\mathcal{G}(X)$-generated.

Theorem 1 *Let X be strongly $\mathcal{G}(X)$-generated and γ the corresponding Grothendieck measure. Let $(f_n)_{n\in\mathbb{N}}\subset\mathcal{L}_1(\mu,X)$ be bounded and uniformly integrable. Then the function*

$$\varphi:\Omega\to[0,\infty],\quad \omega\mapsto\gamma(\{f_n(\omega):n\in\mathbb{N}\})$$

is measurable and

$$\gamma\Big(\Big\{\int_\Omega f_n(\omega)\mu(d\omega):n\in\mathbb{N}\Big\}\Big)\leq\int_\Omega\gamma(\{f_n(\omega):n\in\mathbb{N}\})\mu(d\omega).$$

Proof: Let $G_0\in\mathcal{G}(X)$ be generating. Then

$$\{\varphi<t\}=\bigcup_{s\in[0,t)\cap\mathbb{Q}}\bigcup_{m\in\mathbb{N}}\bigcap_{n\in\mathbb{N}}\{f_n\in mG_0+sB(X)\}$$

for $t>0$ by definition of a generating set. Hence φ is measurable.

To prove the claimed estimate, let us first consider the special case that $S=\sup_{n\in\mathbb{N}}\|f_n\|_\infty<\infty$. Define

$$H(\omega)=\{f_n(\omega):n\in\mathbb{N}\}\subset X\quad\text{and}\quad h(\omega)=\sup_{n\in\mathbb{N}}|f_n(\omega)|\in[0,\infty]$$

for $\omega\in\Omega$. Then $\int_\Omega h(\omega)\mu(d\omega)\leq S\mu(\Omega)<\infty$. Furthermore, since G_0 is a generating set, all measures $\varphi(\omega)=\gamma(H(\omega))$ are determined (up to arbitrary small $\varepsilon>0$) by the ω-independent countable system $\mathcal{E}=\{A_{m,s}:m\in\mathbb{N},s\in\mathbb{Q}\}$ with $A_{m,s}=mG_0+sB(X)$. To be more precise, let $\varepsilon>0$ and $\omega\in\Omega$ be given. According to the definition of γ we find $G\in\mathcal{G}(X)$ such that $H(\omega)\subset G+\varepsilon_1 B(X)$, where $\varepsilon_1=\gamma(H(\omega))+\varepsilon/2$. Choose $\varepsilon_2\in(0,\varepsilon/2]$ with $s=\varepsilon_1+\varepsilon_2\in\mathbb{Q}$. Then $G\subset mG_0+\varepsilon_2 B(X)$ for some $m\in\mathbb{N}$, hence $H(\omega)\subset A_{m,s}$ as well as $\gamma(A_{m,s})\le s\le\gamma(H(\omega))+\varepsilon$.

Therefore we have shown that (3.1) and a condition analogous to (CC) from [11, Theorem 3.1] are satisfied. Now note that the proof of this theorem does not rely on special properties of the Kuratowski or the Hausdorff measure of noncompactness (only $\gamma(A+B)\le\gamma(A)+\gamma(B)$ and $\gamma(\overline{\mathrm{co}}A)=\gamma(A)$ are used). Hence, by writing $\mathcal{E}=\{V_k:k\in\mathbb{N}\}$, we can define $R_0(\omega)=h(\omega)$,

$$R_k(\omega)=\begin{cases}\gamma(V_k) & \text{if } H(\omega)\subset V_k\\ \infty & \text{otherwise}\end{cases}\quad\text{and}\quad r_k(\omega)=\min_{0\le i\le k}R_i(\omega)$$

for $\omega\in\Omega$ and $k\in\mathbb{N}$ to copy the proof from [11] word for word.

Suppose now that the f_n are only uniformly integrable, and let $\varepsilon>0$ be given. Then we find $\delta>0$ such that $\sup_{n\in\mathbb{N}}\int_A|f_n|d\mu\le\varepsilon$ for all $A\in\Sigma$ with $\mu(A)\le\delta$. Since $(f_n)\subset\mathcal{L}_1(\mu,X)$ is bounded, there is a $K>0$ with $\mu(\{|f_n|>K\})\le\delta$ for $n\in\mathbb{N}$. Consequently,

$$\int_\Omega f_n d\mu\in\int_\Omega g_n d\mu+\varepsilon B(X),\quad\text{where}\quad g_n=f_n\mathbf{1}_{\{|f_n|\le K\}}.$$

Because the first part may be applied to (g_n), the claimed estimate follows from $\{g_n(\omega):n\in\mathbb{N}\}\subset\{0\}\cup\{f_n(\omega):n\in\mathbb{N}\}$. □

3 Integral inequalities in special cases

In this section we shall investigate some special cases and variants on Theorem 1 in greater detail. Our first objective is to prove formula (2) from the introduction.

If $\mathcal{G}(X)=\mathcal{K}(X)$, then Lemma 1 and Remark 1(a) together with Theorem 1 imply

Corollary 1 *Let X be separable and $(f_n)_{n\in\mathbb{N}}\subset\mathcal{L}_1(\mu,X)$ bounded and uniformly integrable. Then*

$$\beta\Big(\Big\{\int_\Omega f_n(\omega)\mu(d\omega):n\in\mathbb{N}\Big\}\Big)\le\int_\Omega\beta(\{f_n(\omega):n\in\mathbb{N}\})\mu(d\omega).$$

Corollary 1 is (slightly) more general than [16, Proposition 3] or [11, Corollary 3.1(a)]. We shall see below how this may be used for operator-valued functions $U:\Omega\to L(X,Y)$.

Since we want to show (2), we first have to worry about the measurability of $\omega\mapsto c(U(\omega))$. In fact, we have

Example 2 *Let (Ω,Σ,μ) be the Lebesgue space with $\Omega=[0,1]$. Choose a non-measurable $I\subset[0,1]$ and define $\Gamma=I\times\mathbb{N}$ as well as $X=\ell_1(\Gamma)$. Then, for every infinite-dimensional Banach space Y, there is a strongly measurable $U:\Omega\to L(X,Y)$ such that $\omega\mapsto c(U(\omega))$ is not measurable.*

Proof: For $(s,n)\in\Gamma$ let $e_{(s,n)}$ denote the corresponding unit vector in X. Choose a sequence $(y_n)\subset S(Y)$ such that $\|y_n-y_m\|\geq 1$ for $n\neq m$ and define $U(t)(e_{(s,n)})=\delta_{t,s}y_n$ for $t,s\in\Omega$ and $n\in\mathbb{N}$, with Kronecker's δ. Then $U:\Omega\to L(X,Y)$ is strongly measurable, since $U(\cdot)(e_{(s,n)})=0$ a.e. in Ω for every $(s,n)\in\Gamma$, hence $U(\cdot)(x)=0$ a.e. in Ω for every $x\in X$ by definition of $l_1(\Gamma)$. Moreover, $U(t)=0$ whenever $t\notin I$, and therefore $c(U(t))=0$ for such t. On the other hand, if $t\in I$, then $\{y_n:n\in\mathbb{N}\}\subset U(t)(B(X))$. Consequently, by Remark 1(a) we obtain $2c(U(t))\geq\beta(\{y_n:n\in\mathbb{N}\})\geq 1/2$. Hence $c(U(t))\geq 1/4$ for $t\in I$. Thus, since I is not measurable, $c(U(\cdot))$ is not measurable, too. □

The situation is more favourable in the case that X^* is separable.

Lemma 3 *Let X^* be separable and $U:\Omega\to L(X,Y)$ strongly measurable and bounded. Then $\omega\mapsto c(U(\omega))$ is measurable.*

Proof: By [14, Proposition 3.4] we have for $T\in L(X,Y)$

$$c(T)=\inf\sup\{\|T(x)\|:\ \|x\|=1,|\langle x,z_i^*\rangle|<\epsilon\},$$

where the infimum is taken over all finite collections $z_1^*,\dots,z_l^*\in B(X^*)$ and $\epsilon>0$. Choose $(x_n^*)_{n\in\mathbb{N}}\subset B(X^*)$ dense and let $W_{k,N}=\{x\in X:|\langle x,x_i^*\rangle|<k^{-1}\,\text{for}\,i\in N\}$ for all $k\in\mathbb{N}$ and finite $N\subset\mathbb{N}$. Then

$$c(T)=\inf_{(k,N)}\ \sup\{\|T(x)\|:\ \|x\|=1,x\in W_{k,N}\}, \tag{3}$$

because $(x_n^*)_{n\in\mathbb{N}}\subset B(X^*)$ is dense. Since X is also separable, $\omega\mapsto\sup\{\|U(\omega)(x)\|:\|x\|=1,x\in W_{k,N}\}$ is measurable for a strongly measurable and bounded $U:\Omega\to L(X,Y)$. Hence the claim follows from (3), since the infimum is a countable one. □

Nevertheless, Lemma 2 shows that in general the right-hand side of (2) has to be considered as an upper integral; recall the following

Definition 4 *Let $f:\Omega\to\mathbb{R}$ be bounded (but not necessarily measurable). Then the upper integral of f is*

$$\overline{\int_\Omega} f(\omega)\mu(d\omega)=\inf\Big\{\int_\Omega g(\omega)\mu(d\omega):g\in\mathcal{L}_1(\mu),\ g(\omega)\geq f(\omega)\ \text{in}\,\Omega\Big\}.$$

To prepare for the next lemma, we shall introduce a further class of Banach spaces.

Definition 5 *A Banach space X has property $\mathcal{S}$ if for all separable subspaces $X_0\subset X$ there exist a separable subspace Z of X with $X_0\subset Z$ and a projection $P:X\to Z$ of norm 1.*

In particular, WCG Banach spaces have property $\mathcal{S}$; cf. [2]. We shall also need some estimates for β and β restricted to subspaces; here we let $\beta_{X_0}(A)=\inf\{r>0:A\subset K+r\,B(X_0)$ for some compact $X_0\}$, if X_0 is a subspace of X and $A\subset X_0$ is bounded.

Lemma 4 *The following assertions hold:*

(a) $\beta(A)\leq 2\sup_{A_0\subset A}\beta(A_0)$ for bounded $A\subset X$, where the "sup" is taken over all countable subsets A_0 of A;

(b) $\beta(A)\le\beta_{X_0}(A)\le 2\beta(A)$ *for bounded* $A\subset X_0$*;*

(c) if X *has property* $\mathcal{S}$*,* $X_0\subset X$ *is a separable subspace, and* Z *is choosen according to the definition of property* $\mathcal{S}$*, then* $\beta_Z(A)=\beta(A)$ *for bounded* $A\subset Z$*.*

Proof: (a) Let $r=\beta(A)>0$ and $\varepsilon\in(0,r)$ be given. For fixed $x_1\in A$ we find an $x_2\in A$ such that $\|x_2-x_1\|>r-\varepsilon$, and inductively a sequence (x_n) with $\|x_n-x_m\|>r-\varepsilon$ for $n\neq m$. Hence, defining $A_1=\{x_n:n\in\mathbb{N}\}$ we obtain $2\sup_{A_0\subset A}\beta(A_0)\ge 2\beta(A_1)\ge r-\varepsilon$. This shows (a), and (b) resp. (c) are equation (4) resp. Proposition 2 (iii) from [16]. □

Now we can prove

Theorem 2 *Let* $U:\Omega\to L(X,Y)$ *be strongly measurable and bounded.*

(a) If X^* *is separable, then*

$$c\Big(\int_\Omega U(\omega)\mu(d\omega)\Big)\le\int_\Omega c(U(\omega))\mu(d\omega).$$

(b) If X *is separable, then*

$$c\Big(\int_\Omega U(\omega)\mu(d\omega)\Big)\le 2\cdot\overline{\int_\Omega} c(U(\omega))\mu(d\omega).$$

(c) If X *and* Y *are arbitrary, then*

$$c\Big(\int_\Omega U(\omega)\mu(d\omega)\Big)\le 4\cdot\overline{\int_\Omega} c(U(\omega))\mu(d\omega).$$

Proof: Let $T=\int_\Omega U\,d\mu$. Concerning (a) we first note that $\|T(x)\|\le\int_\Omega\|U(\cdot)(x)\|d\mu$ for $x\in X$, since U is strongly measurable. Thus

$$\sup\{\|T(x)\|:\ \|x\|=1,x\in W_{k,N}\}\le\int_\Omega\sup\{\|U(\cdot)(x)\|:\ \|x\|=1,x\in W_{k,N}\}\,d\mu$$

for all $k\in\mathbb{N}$ and finite $N\subset\mathbb{N}$, cf. the proof to Lemma 3. Evidently for $S\in L(X,Y)$, $k,l\in\mathbb{N}$ and finite $N,M\subset\mathbb{N}$ with $l\le k$ and $M\subset N$ we have

$$\sup\{\|S(x)\|:\ \|x\|=1,x\in W_{k,N}\}\le\sup\{\|S(x)\|:\ \|x\|=1,x\in W_{l,M}\}.$$

Consequently,

$$\begin{aligned} c(T) &= \inf_{(k,N)}\sup\{\|T(x)\|:\|x\|=1,x\in W_{k,N}\}\\ &\le \inf_{(k,N)}\int_\Omega\sup\{\|U(\cdot)(x)\|:\|x\|=1,x\in W_{k,N}\}\,d\mu\\ &= \int_\Omega\inf_{(k,N)}\sup\{\|U(\cdot)(x)\|:\|x\|=1,x\in W_{k,N}\}\,d\mu\\ &= \int_\Omega c(U(\cdot))\,d\mu. \end{aligned}$$

To see (b), let $A = T(B(X)) \subset Y$. Since X separable and U is strongly measurable, we find a μ-null set $N \subset \Omega$ and a separable subspace Y_0 of Y such that $\{U(\omega)x : \omega \in \Omega \setminus N, x \in X\} \subset Y_0$. W.l.o.g. we may assume $N = \emptyset$ because T is an integral. Now we can apply Corollary 1 to Y_0 and $f_n = U(\cdot)(x_n)$ with $(x_n) \subset B(X)$ dense. Therefore Remark 1(a) and Lemma 4(b) yield

$$\begin{aligned} c(T) &\leq \beta(A) \leq \beta_{Y_0}(A) = \beta_{Y_0}\Big(\Big\{\int_\Omega f_n(\omega)\mu(d\omega) : n \in \mathbb{N}\Big\}\Big) \\ &\leq \int_\Omega \beta_{Y_0}(U(\omega)(B(X))\mu(d\omega) \leq 2\int_\Omega c_{Y_0}(U(\omega))\mu(d\omega) = 2\int_\Omega c(U(\omega))\mu(d\omega). \end{aligned}$$

For the proof of (c) fix $\varepsilon \in (0, \beta(A))$ and a countable $A_0 \subset A$ such that $\beta(A) \leq 2\beta(A_0) + \varepsilon$. Then $A_0 = \{Tx_n : n \in \mathbb{N}\}$ for some $(x_n) \subset B(X)$. Choose separable subspaces $X_0 \subset X$, $Y_0 \subset Y$ and a μ-null set $N \subset \Omega$ such that $(x_n) \subset B(X_0)$ as well as $\{U(\omega)x : \omega \in \Omega \setminus N, x \in X_0\} \subset Y_0$; again we can assume that $N = \emptyset$. Now we may proceed analogously to (a) by applying Theorem 1 to Y_0. □

As already mentioned in the introduction, (b) is-up to a constant-the formula claimed in [26, Proposition 2.2]. This provides us with an easy proof of the following corollary, i.e. of [24, Theorem 1.3].

Corollary 2 *Let $U : \Omega \to K(X,Y)$ be strongly measurable and bounded. Then the strong integral $\int_\Omega U d\mu$ is again compact.*

By using Theorem 1, preliminary results of this type may also be obtained for Grothendieck classes $\mathcal{G}(X)$ different from $\mathcal{K}(X)$. For, let $D_\infty(X,Y)$ resp. $BS(X,Y)$ denote the Asplund operators resp. the Banach-Saks operators, i.e. the $T \in L(X,Y)$ such that $T(B(X)) \subset Y$ is an Asplund set resp. a Banach-Saks set.

Corollary 3 *Let X be separable and Y strongly Asplund generated. If $U : \Omega \to D_\infty(X,Y)$ is strongly measurable and bounded, then $\int_\Omega U d\mu$ is again an Asplund-operator.*

Proof: Let $\gamma_{\mathcal{A}}$ denote the Grothendieck measure for $\mathcal{G}(Y) = \mathcal{A}(Y)$. Then we have $\gamma_{\mathcal{A}}(T(B(X))) = \gamma_{\mathcal{A}}(\{Tx_n : n \in \mathbb{N}\})$, where $T = \int_\Omega U d\mu$ and $(x_n) \subset B(X)$ is dense. Therefore we may apply Theorem 1 with $f_n : \Omega \to Y$ defined as $f_n(\omega) = U(\omega)(x_n)$. □

The same proof gives

Corollary 4 *Let X be separable and Y strongly Banach-Saks generated. If $U : \Omega \to BS(X,Y)$ is strongly measurable and bounded, then $\int_\Omega U d\mu$ is again a Banach-Saks operator.*

Remark 2 *(a) The results corresponding to Corollaries 2, 3 or 4 for weakly compact resp. conditionally weakly compact operators are true for all Banach spaces X and Y; cf. [19].*

(b) Generalizations are possible to the case that (Ω, Σ, μ) is not finite, but localizable, and a certain uniform integrability conditions holds. This can be seen analogously to [18].

Sometimes it may be useful to know the formula from Corollary 1 also under relaxed assumptions on X or in the case of an uncountable number of functions.

Theorem 3 *Let I be an index set and $(f_\tau)_{\tau\in I}\subset\mathcal{L}_1(\mu,X)$ bounded and uniformly integrable.*

(a) If I is countable and X has property $\mathcal{S}$, then

$$\beta\Big(\Big\{\int_\Omega f_\tau(\omega)\mu(d\omega):\tau\in I\Big\}\Big)\le\int_\Omega\beta(\{f_\tau(\omega):\tau\in I\})\mu(d\omega).$$

(b) If X has property $\mathcal{S}$, then

$$\beta\Big(\Big\{\int_\Omega f_\tau(\omega)\mu(d\omega):\tau\in I\Big\}\Big)\le 2\cdot\overline{\int_\Omega}\beta(\{f_\tau(\omega):\tau\in I\})\mu(d\omega).$$

(c) If X is arbitrary, then

$$\beta\Big(\Big\{\int_\Omega f_\tau(\omega)\mu(d\omega):\tau\in I\Big\}\Big)\le 4\cdot\overline{\int_\Omega}\beta(\{f_\tau(\omega):\tau\in I\})\mu(d\omega).$$

Proof: The proof may be carried out analogous to the proof of Theorem 2. Note that the factor 2 in (b) is picked up by reducing to the countable case by means of Lemma 4(a). Since X has property $\mathcal{S}$, Lemma 4(c) implies that no additional factor is needed. On the other hand, in (c) we also have to use $\beta_{X_0}(A)\le 2\beta(A)$ from Lemma 4(b). □

Theorem 3(a) is (slightly) more general than [15, Proposition 1.6.(i)].

Integral inequalities like Theorem 3(c) are difficult to prove for Grothendieck classes $\mathcal{G}(X)$ being different from $\mathcal{K}(X)$, at least if X is not supposed to be strongly $\mathcal{G}(X)$-generated. For the demonstration of further results without additional assumptions on X and modified hypothesis on $M=\{f_\tau:\tau\in I\}$, the following concept will turn out to be useful.

Definition 6 *Let $\mathcal{G}(X)\subset\mathcal{B}(X)$ be a Grothendieck class and $M\subset\mathcal{L}_1(\mu,X)$ bounded. A measurable $\phi:\Omega\to[0,\infty]$ is called admissible, if for $\omega\in\Omega$ there exists a set $G(M,\phi,\omega)\in\mathcal{G}(X)$ such that*

(a) $\{f(\omega):f\in M\}\subset G(M,\phi,\omega)+\phi(\omega)B(X)$ for $\omega\in\Omega$, where we let $\phi(\omega)B(X)=X$ if $\phi(\omega)=\infty$;

(b) for every $f\in M$ there is a measurable $T_f:\Omega\to X$ with $T_f(\omega)\in G(M,\phi,\omega)$ and $\|f(\omega)-T_f(\omega)\|\le\phi(\omega)$ for $\omega\in\Omega$.

It may happen that $\phi=\infty$ is the only admissible function. This follows from (a) in the simple example $X=\mathbb{R}$, $\mathcal{G}(\mathbb{R})=\mathcal{B}(\mathbb{R})$, $\Omega=[0,1]$, and $M=\{f_{i,j}:(i,j)\in\mathbb{N}\times\mathbb{N},1\le i\le j\}$, where $f_{i,j}=j\cdot\mathbf{1}_{[(i-1)/j,\,i/j]}$, since $\|f_{i,j}\|_1=1$, but $\sup_{i,j}|f_{i,j}(t)|=\infty$ for every $t\in[0,1]$.

Note that $\phi_1\wedge\phi_2$ is admissible if ϕ_1 and ϕ_2 are. Since the class of admissible functions is nonempty, we may define $\gamma^0:\Omega\to[0,\infty]$ by

$$\gamma^0=\inf\{\phi:\phi\text{ is admissible}\},$$

where the infimum is taken in the order-lattice of measurable functions as equivalence classes. For notational convenience the dependence of γ^0 on $\mathcal{G}(X)$, Ω and M will be supressed exept when we want to emphasize this dependence. By definition, γ^0 is measurable with

$$\int_\Omega\gamma^0\,d\mu=\inf\Big\{\int_\Omega\phi\,d\mu:\phi\text{ is admissible}\Big\}.$$

Moreover, if γ denotes the Grothendieck measure associated with $\mathcal{G}(X)$, we have $\gamma(\{f(\cdot): f\in M\})\le\gamma^0$ a.e. in Ω.

Our next result shows that it is worth to look for conditions that ensure $\gamma(\{f(\cdot):f\in M\})=\gamma^0$, at least in the special cases $\mathcal{G}(X)=\mathcal{K}(X),\mathcal{G}(X)=\mathcal{W}(X)$ resp. $\mathcal{G}(X)=\mathcal{CW}(X)$ with corresponding Grothendieck measures β,γ_w resp. γ_{cw}.

Theorem 4 *Let $M\subset\mathcal{L}_1(\mu,X)$ be bounded and uniformly integrable. If $\gamma=\beta$, $\gamma=\gamma_w$ or $\gamma=\gamma_{cw}$, then*

$$\gamma\Big(\Big\{\int_\Omega f(\omega)\mu(d\omega):f\in M\Big\}\Big)\le\int_\Omega\gamma^0(\omega)\mu(d\omega).$$

Proof: Fix an admissible $\phi:\Omega\to[0,\infty)$. We may assume $\phi\in\mathcal{L}_1(\mu)$, since we have to show $\gamma(\{\int_\Omega f d\mu:f\in M\})\le\int_\Omega\phi d\mu$. By definition of an admissible function,

$$\Big\{\int_\Omega f d\mu:f\in M\Big\}\subset\Big\{\int_\Omega T_f d\mu:f\in M\Big\}+\Big(\int_\Omega\phi d\mu\Big)B(X).$$

Therefore it is enough to prove $\{\int_\Omega T_f d\mu:f\in M\}\in\mathcal{G}(X)$. Because of $\{T_f(\omega):f\in M\}\subset G(M,\phi,\omega)\in\mathcal{G}(X)$ for a.a. $\omega\in\Omega$, it follows that $\{T_f(\omega):f\in M\}\in\mathcal{G}(X)$ for a.a. $\omega\in\Omega$. Since $\{T_f:f\in M\}\subset\mathcal{L}_1(\mu,X)$ is bounded and uniformly integrable, we may use [13, Lemma 2.6] for $\mathcal{G}(X)=\mathcal{K}(X)$ resp. [19, Remark 2.4] for $\mathcal{G}(X)=\mathcal{W}(X)$ or $\mathcal{G}(X)=\mathcal{CW}(X)$ to obtain $\{\int_\Omega T_f d\mu:f\in M\}\in\mathcal{G}(X)$ in those cases. $\square$

Under some circumstances we may prove $\gamma(\{f(\cdot):f\in M\})=\gamma^0$. The first result of this type shows that Theorem 4 is compatible with Theorem 1.

Lemma 5 *Let X be strongly $\mathcal{G}(X)$-generated and γ the corresponding Grothendieck measure. Let $M\subset\mathcal{L}_1(\mu,X)$ be countable, bounded and uniformly integrable. Then $\varphi(\cdot)=\gamma(\{f(\cdot):f\in M\})=\gamma^0$.*

Proof: From Theorem 1 we know that φ is measurable. To obtain $\gamma^0\le r$ on $A_r=\{\varphi\le r\}$ for every $r\ge 0$, we fix an $\varepsilon>0$, a generating set $G_0\in\mathcal{G}(X)$ and $N\in\Sigma$ with $\mu(N)=0$ such that $S=\{f(\omega):\omega\in\Omega\setminus N,f\in M\}$ is separable. It is enough to show that for $r\ge 0$ with $A_r\neq\emptyset$ the function $\phi(\omega)=r+\varepsilon$ on $A_r\setminus N$, $\phi(\omega)=\infty$ on $A_r^c\cup N$, is admissible. For, by definition of a generating set and by definition of φ we have

$$A_r\setminus N=\bigcup_{n\in\mathbb{N}}A^n\quad\text{with}\quad A^n=\{\omega\in A_r\setminus N:M(\omega)\subset nG_0+(r+\varepsilon/2)B(X)\},$$

where $M(\omega)=\{f(\omega):f\in M\}$, hence a disjoint union

$$A_r\setminus N=\bigcup_{n\in\mathbb{N}}B^n\quad\text{with}\quad B^1=A^1\quad\text{and}\quad B^n=A^n\setminus\bigcup_{j=1}^{n-1}A^j\quad\text{for}\quad n\ge 2.$$

Therefore we can choose $G(M,\phi,\omega)=nG_0$ for $\omega\in B^n$ and $G(M,\phi,\omega)=\{0\}$ for $\omega\in A_r^c\cup N$. Moreover, to find T_f for $f\in M$ fixed, note that

$$M(B^n)=\bigcup_{\omega\in B^n}M(\omega)\subset S\cap(nG_0+(r+\varepsilon/2)B(X))$$

for every $n\in\mathbb{N}$. This implies that there exist $(x_j^n)_{j\in\mathbb{N}}\subset M(B^n)$ dense and $(y_j^n)_{j\in\mathbb{N}}\subset G_0$ such that $\|x_j^n-ny_j^n\|\le r+\varepsilon/2$ for all $n,j\in\mathbb{N}$. If we let

$$C_j^n(f)=\{\omega\in B^n:\|f(\omega)-ny_j^n\|\le r+\varepsilon\}\,,\quad \text{then}\quad B^n=\bigcup_{j\in\mathbb{N}}C_j^n(f)$$

for every $n\in\mathbb{N}$. Hence we have again a disjoint union

$$B^n=\bigcup_{j\in\mathbb{N}}D_j^n(f)\quad\text{with}\quad D_1^n(f)=C_1^n(f)\quad\text{and}\quad D_j^n(f)=C_j^n(f)\setminus\bigcup_{i=1}^{j-1}C_i^n(f)$$

for $j\ge 2$. Therefore we can let $T_f(\omega)=ny_j^n$ on $D_j^n(f)$ and $T_f(\omega)=0$ on $A_r^c\cup N$ to see that ϕ is admissible. □

If the family M is even uniformly measurable, we may also show $\gamma(\{f(\cdot):f\in M\})=\gamma^0$. To this end let us first recall the concept of uniform measurability.

Definition 7 *Let $M\subset\mathcal{L}_1(\mu,X)$ be bounded. M is said to be uniformly measurable, if for all $\varepsilon>0$ and $A\in\Sigma$ there exists an $A_\varepsilon\in\Sigma$ and $A_1,\dots,A_n\subset A_\varepsilon$ pairwise disjoint with $\mu(A\setminus A_\varepsilon)<\varepsilon$ and $\cup_{i=1}^n A_i=A_\varepsilon$ such that for all $i=1,\dots,n$ we may choose $\omega_i\in A_i$ satisfying*

$$\sup_{f\in M}\|f\,\mathbf{1}_{A_\varepsilon}-\sum_{i=1}^n f(\omega_i)\,\mathbf{1}_{A_i}\|_\infty<\varepsilon\,.$$

Example 3 *M is uniformly measurable if*

(a) X is a Schur space and $M\subset L_\infty(\mu,X)$ is relatively weakly compact (cf. [3] resp. [20, Theorem 2.6]) or

(b) Ω is compact and $M\subset C(\Omega,X)$ is equicontinuous.

Lemma 6 *Let $M\subset\mathcal{L}_1(\mu,X)$ be countable, bounded and uniformly measurable. Then $\gamma(\{f(\cdot):f\in M\})=\gamma^0$.*

Proof: Retaining the notation from Lemma 5, we first remark that for given $\varepsilon>0$ and $A\in\Sigma$ we find $G_\varepsilon\in\mathcal{G}(X)$ and $A_\varepsilon\in\Sigma$ such that $A_\varepsilon\subset A$, $\mu(A\setminus A_\varepsilon)<\varepsilon$ and $M(\omega)\subset G_\varepsilon+(\varphi(\omega)+\varepsilon)B(X)$ for $\omega\in A_\varepsilon$. Indeed, we may choose $A_\varepsilon\in\Sigma$, A_i and ω_i according to the definition of uniform measurability. Then $M(\omega_i)\subset G_i+(\varphi(\omega_i)+\varepsilon)B(X)$ for some $G_i\in\mathcal{G}(X)$. Hence the claim follows with $G_\varepsilon=\cup_{i=1}^n G_i$, since $\omega\in A_i$ implies $\varphi(\omega_i)\le\varphi(\omega)+\varepsilon$. We also can achieve $A_\varepsilon\subset A$ by replacing A with $A\cap A_\varepsilon$ if necessary.

Now we may proceed as in the proof to Lemma 5, since for $A=A_r=\{\varphi\le r\}\in\Sigma$ and fixed $\varepsilon>0$ we may apply the above remark to all $n\in\mathbb{N}$ with $n>1/\varepsilon$ and $\varepsilon_n=1/2n$ to find $G^n\in\mathcal{G}(X)$ and $A^n\subset A$ with $\mu(A\setminus A^n)<\varepsilon_n$ and $M(\omega)\subset G^n+(r+\varepsilon/2)B(X)$ on A^n, hence

$$A\setminus N_1=\bigcup_{n\in\mathbb{N}}\{\omega\in A\setminus N_1:M(\omega)\subset G^n+(r+\varepsilon/2)B(X)\}$$

where $N_1=A\setminus\bigcup_{n\in\mathbb{N}}A^n$ has $\mu(N_1)=0$. □

Therefore from Theorem 4 and Lemma 6 we obtain

Corollary 5 *Let $M\subset\mathcal{L}_1(\mu,X)$ be countable, bounded and uniformly measurable. If $\gamma=\beta$, $\gamma=\gamma_w$ or $\gamma=\gamma_{cw}$, then*

$$\gamma\Big(\Big\{\int_\Omega f(\omega)\mu(d\omega):f\in M\Big\}\Big)\le\int_\Omega\gamma(\{f(\omega):f\in M\})\mu(d\omega).$$

Besides giving a new condition under which an integral inequality like Corollary 5 is valid, in the case $\mathcal{G}(X)=\mathcal{W}(X)$ Theorem 4 may be used to obtain a generalization resp. a simplified proof of a result concerning weak solutions of differential equations in Banach spaces; cf. [17] and Corollary 6 below, and also [8, Lemma 18.3] for a protoype of applying this technique.

Theorem 5 *Let $b>0$ and $f:[0,b]\times X\to X$ be weakly continous with $\sup_{(t,x)\in[0,b]\times X}\|f(t,x)\|=C<\infty$. Suppose that $\omega:[0,b]\times[0,\infty)\to[0,\infty)$ is a Kamke function, i.e. $\omega(\cdot,y)$ is measurable, $\omega(t,\cdot)$ is continuous, $\omega(t,y)\le\Phi(t)$ for some integrable Φ, and $\varphi(t)\le\int_0^t\omega(s,\varphi(s))ds,\varphi(0)=0$, implies $\varphi(t)=0$ in $[0,b]$. If*

- *(a) X is SWCG and $\gamma_w(f(t,A))\le\omega(t,\gamma_w(A))$ for $t\in[0,b]$ and bounded $A\subset X$, or*
- *(b) for all uniformly Lipschitz continuous sequences $(v_n)\subset C([0,b],X)$ and all $t\in[0,b]$ we have*
$$\gamma^0_{\mathcal{W}(X),\Omega,M}(s)\le\omega(s,\gamma_w(\{v_n(s):n\in\mathbb{N}\}))\quad a.e.\ in\quad[0,t],$$
with $\Omega=[0,t]$ and $M=\{f(\cdot,v_n(\cdot)):n\in\mathbb{N}\}$,

then, for every $x_0\in X$, the initial value $x'=f(t,x),x(0)=x_0$, has a solution (which is understood to be strongly continuous and to have a weak derivative which satisfies the differential equation).

Proof: For every $n\in\mathbb{N}$ with $n\ge 1/b$ and $t\in[0,b]$ the Peano approximations

$$u_n(t)=\begin{cases} x_0 & : \ t\in[0,1/n]\\ x_0+\displaystyle\int_0^{t-1/n}f(s,u_n(s))\,ds & : \ t\in[1/n,b]\end{cases}$$

are defined and $\sup_{n\in\mathbb{N}}\|u_n(s)-u_n(t)\|\le C\|s-t\|$ for $s,t\in[0,b]$. Hence, to find $u\in C([0,b],X)$ with $u(t)=x_0+\int_0^t f(s,u(s))\,ds$ on $[0,b]$, i.e. a weak solution, we only have to show that $\varphi(t)=\gamma_w(\{u_n(t):n\in\mathbb{N}\})=0$ on $(0,b]$ by the "weak" Arzelà-Ascoli theorem. For, fix $t\in(0,b)$ and $p\in\mathbb{N}$ with $p\ge 1/t$. Then Lemma 2 and Theorem 4, applied to $\mathcal{G}(X)=\mathcal{W}(X)$, $\Omega=[0,t]$ and $M=\{f(\cdot,u_n(\cdot)):n\ge p\}$, yield

$$\begin{aligned}\varphi(t)&=\gamma_w(\{u_n(t):n\ge p\})\\ &\le\gamma_w\Big(\Big\{\int_0^{t-1/n}f(s,u_n(s))\,ds:n\ge p\Big\}\Big)\\ &\le\gamma_w((C/p)\,B(X))+\gamma_w\Big(\Big\{\int_0^t f(s,u_n(s))\,ds:n\ge p\Big\}\Big)\\ &\le C/p+\int_0^t\gamma^0_{\mathcal{W}(X),\Omega,M}(s)\,ds.\end{aligned}$$

Concerning (a), note that $\gamma^0_{\mathcal{W}(X),\Omega,M}=\gamma_w(\{f(\cdot,u_n(\cdot):n\geq p\})$ by Lemma 5. Hence in both cases (a) or (b)

$$\varphi(t)\leq C/p+\int_0^t \omega(s,\gamma_w(\{u_n(s):n\geq p\}))\,ds$$

for all $t\in(0,b)$ and $p\in\mathbb{N}$ with $p\geq 1/t$. Therefore the claim follows from $\varphi(0)=0$, since ω is a Kamke function. □

An easy consequence of the above theorem is the main result of [17].

Corollary 6 *In the setting of Theorem 5, (b) is satisfied in particular, if*

$$\lim_{h\to 0^+}\gamma_w(f([t,t+h]\times A))\leq\omega(t,\gamma_w(A)) \tag{4}$$

for all $t\in[0,b)$ and bounded $A\subset X$. Hence, for every $x_0\in X$, the initial value problem $x'=f(t,x),x(0)=x_0$, has a solution in this case.

Proof: Fix a uniformly Lipschitz continuous sequence $(v_n)\subset C([0,b],X)$. Let us go ahead to prove that (4) implies $\gamma^0(s)\leq\omega(s,\gamma_w(A(s)))$ for a.a. $s\in(0,t)$, as we have to show for (b). Here we defined $A(s)=\{v_n(s):n\in\mathbb{N}\}$. Since $\omega(s,\cdot)$ is continuous and $\varphi(\cdot)=\gamma_w(A(\cdot))$ is Lipschitz, it is sufficient to verify that for $s\in(0,t)$ and all small $\varepsilon>0$ there exists an $h=h(s,\varepsilon)>0$ such that $\gamma^0(\tau)\leq\omega(s,\gamma_w(A([s,s+\varepsilon])))+\varepsilon=r(s,\varepsilon)+\varepsilon$ for a.a. $\tau\in[s,s+h]$, where we let $A([s,s+\varepsilon])=\{v_n(\tau):\tau\in[s,s+\varepsilon],n\in\mathbb{N}\}$. But this follows almost analogously to the proof of Theorem 5: Because every function in M is measurable, we find $N\in\Sigma$ with $\mu(N)=0$ such that $S=\{f(\tau,v_n(\tau)):\tau\in[0,t]\setminus N,n\in\mathbb{N}\}$ is separable. Moreover,

$$f([s,s+h]\times A([s,s+\varepsilon]))\subset W+(r(s,\varepsilon)+\varepsilon/2)\,B(X)$$

for some $h\in(0,\varepsilon]$ with $s+h\leq t$ and $W\in\mathcal{W}(X)$, by (4). Hence we only have to prove that $\phi(\tau)=r(s,\varepsilon)+\varepsilon$ on $J=[s,s+h]\setminus N$, $\phi(\tau)=\infty$ on $[0,t]\setminus J$, defines an admissible function. To see this, let $G(M,\phi,\tau)=W$ for $\tau\in J$ and $G(M,\phi,\tau)=\{0\}$ for $\tau\in[0,t]\setminus J$. If $u\in M$ is fixed, then we may choose $(x_j)\subset u(J)\subset S$ dense, and therefore $(w_j)\subset W$ such that $\|x_j-w_j\|\leq r(s,\varepsilon)+\varepsilon/2$ for $j\in\mathbb{N}$. Then

$$J=\bigcup_{j\in\mathbb{N}}\{\tau\in J:\|u(\tau)-w_j\|\leq r(s,\varepsilon)+\varepsilon\},$$

and consequently we may define $T_u(\tau)=w_j$ on I_j (where (I_j) is the corresponding disjoint partition of J) and $T_u(\tau)=0$ on $[0,t]\setminus J$; c.f. the proof to Theorem 5. □

4 Characterizations of strongly $\mathcal{W}(X)$, $\mathcal{A}(X)$, or $\mathcal{CW}(X)$-generated Banach spaces

In this section we shall consider *SWCG*, *SAG* and *SCWG* Banach spaces in greater detail. We start by reviewing some positive and negative examples of such spaces.

Example 4 *(a) SWCG are all reflexive spaces, all separable Schur spaces, the space of operators of trace class on a separable Hilbert space, $L_1(\mu)$ for σ-finite μ, the predual of a σ-finite v. Neumann algebra, the weakly sequentially complete Azimi-Hagler space X_0 from [4], H_1, L_1/H_1; cf. [22].*

Not SWCG are all spaces that are not weakly sequentially complete, the Batt-Hiermeyer space BH from [5], certain vector-valued Lorentz sequence spaces; cf. [22] and [27].

(b) SAG are all Asplund spaces, all SWCG-spaces, ℓ_1- sums of SAG-spaces (this may be proved analogous to [22, Proposition 1.2.]).

Not SAG are all weak sequentially complete spaces which are not SWCG, like the BH-space or $\ell_2(\ell_1)$, furthermore $C([0,1])$ (since $\ell_2(\ell_1)$ embeds into $C([0,1])$ isometrically, cf. [9, p.36]), the James tree JT, the James-Hagler space JH from [10] (proof analogous to [22, Example 2.6.]), ℓ_∞.

(c) SCWG are all spaces not containing ℓ_1 like c_0, JT, JH, all SAG-spaces, ℓ_1-sum of SCWG-spaces.

Not SCWG are all weakly sequentially complete spaces which are not SWCG, $c_0(\ell_1)$ (according to Theorem 9 below), $C([0,1])$ (since $c_0(\ell_1)$ embeds isometrically into $C([0,1])$), ℓ_∞.

In the following characterization of $SWCG$-spaces we denote by τ_W the dual Mackey-topology, i.e. the locally convex topology on X^* which is generated by the polars of the weakly compact sets in X; cf. [12, p.262 (2)].

Theorem 6 *For a Banach space X there are equivalent*

(a) X is SWCG;

(b) $\exists\ R$ reflexive $\exists\ T_0 \in L(R,X)\ \forall\ T \in W(\ell_1,X)\ \forall\ \varepsilon>0\ \exists\ T_\varepsilon \in L(\ell_1,R):\ \|T-T_0\circ T_\varepsilon\|<\varepsilon$;

(c) $(B(X^*),\tau_W)$ is metrizable.

Proof: The equivalence of (a) and (c) is from [22, Thm.2.1]. To prove that (a) implies (b), let $K_0 \subset X$ be an absolutely convex and relatively weakly compact set according to Definition 2. Let $R = X_C$, where $X_C \subset X$ as a set, X_C is reflexive and $\overline{K_0} \subset C = B(X_C)$. The space X_C is constructed according to the scheme of Davis et al. (cf. [9, Lemma 3, p.227]), and the inclusion $T_0 = \imath : X_C \to X$ is continuous. If $T \in W(\ell_1,X)$ and $\varepsilon>0$ are given, then we find $m \in \mathbb{N}$ such that $T(B(\ell_1)) \subset mK_0 + \varepsilon/2B(X)$, hence a sequence $(x_n) \subset K_0$ with $\|Te_n - mx_n\| \le \varepsilon/2$ for all $n \in \mathbb{N}$. Consequently, we can take $T_\varepsilon : \ell_1 \to X_C$, $e_n \mapsto mx_n$.

If (b) holds, then $K_0 = T_0(B(R)) \subset X$ is absolutely convex and weakly compact. Suppose that K_0 is not a generating set. Then we would find a relatively weakly compact $K \subset X$, a sequence $(x_n) \subset K$ and $\varepsilon>0$ with $x_n \notin nK_0 + \varepsilon B(X)$ for all $n \in \mathbb{N}$. Therefore we may define $T \in W(\ell_1,X)$ by $e_n \mapsto x_n$, and since K_0 is absolutely convex we have $z_n = (T_0\circ T_\varepsilon)(e_n/n) \in K_0$ for $n \ge \|T_\varepsilon\|$ with T_ε from (b). But this implies $\|x_n - nz_n\| < \varepsilon$ for n big enough, a contradiction. □

Similar to the Mackey topology, on X^* the locally convex topologies τ_A and τ_{CW} may be defined which are generated by the polars of Asplund resp. conditional weakly compact sets in X. Then, since weakly compact $\Rightarrow$ Asplund $\Rightarrow$ conditionally weakly compact, one has $\tau_W \subset \tau_A \subset \tau_{CW} \subset \|\cdot\|$ on X^*.

By using [7, Lemma 5.3.4] resp. [9, p.237, Ex. 2], both following theorems can be proved in an analogous way to the previous one.

Theorem 7 *For a Banach space X there are equivalent*

(a) X is SAG;

(b) $\exists$ Asplund space Z $\exists\, T_0 \in L(Z,X)$ $\forall\, T \in D_\infty(\ell_1,X)$ $\forall\, \varepsilon > 0$ $\exists\, T_\varepsilon \in L(X,Z)$: $\|T - T_0 \circ T_\varepsilon\| < \varepsilon$;

(c) $(B(X^*),\tau_A)$ is metrizable.

Theorem 8 *For a Banach space X there are equivalent*

(a) X is SCWG;

(b) $\exists\, Z$ not containing ℓ_1 $\exists\, T_0 \in L(Z,X)$ $\forall\, T \in CW(\ell_1,X)$ $\forall\, \varepsilon > 0$ $\exists\, T_\varepsilon \in L(X,Z)$: $\|T - T_0 \circ T_\varepsilon\| < \varepsilon$;

(c) $(B(X^*),\tau_{CW})$ is metrizable.

The result corresponding to the following one for $SWCG$-spaces is [23, Thm.5.10].

Theorem 9 *If $(X_n)_{n\in\mathbb{N}}$ is a sequence of SAG-spaces (resp. SCWG-spaces), then $c_0(X_n)$ and $\ell_p(X_n)$ for $1<p<\infty$ are again SAG-spaces (resp. SCWG-spaces) if and only if all but finitely many of the X_n are Asplund spaces (resp. Banach spaces not containing a copy of ℓ_1).*

Proof: Since the $SCWG$-case may be proved like the SAG-case and the arguments are mainly the same for $\ell_p(X_n)$, we consider $X = c_0(X_n)$ and assume that X is SAG, but X_n is not an Asplund space for infinitely many $n \in T$, where $T \subset \mathbb{N}$ is a subsequence. Choose an absolutely convex generating $K_0 \subset X$ and let $K_n = pr_n(K_0) \subset X_n$ for $n \in \mathbb{N}$, where pr_n is the projection on the n-th coordinate. Then K_n is Asplund. Moreover, for fixed $n \in T$ it is impossible that $B(X_n) \subset nK_n + \frac{1}{2}B(X_n)$, since this would imply $2^{-m}B(X_n) \subset n2^{-m}K_n + 2^{-(m+1)}B(X_n)$ for all $m \in \mathbb{N}$, hence inductively $B(X_n) \subset \bigcap_{m\in\mathbb{N}} (A_m + 2^{-m}B(X_n))$ with Asplund sets $A_m = 2(1-2^{-m})nK_n \subset X_n$. Because $B(X_n)$ in not an Asplund set, this contradicts [7, Lemma 5.3.4]. Therefore we find $x_n \notin nK_n + \frac{1}{2}B(X_n)$ for all $n \in T$. Now let $L = \{(\delta_{ni}x_n)_{i\in\mathbb{N}} : n \in T\} \subset X$. Since $X^* = \ell_1(X_n^*)$, the set $L \subset X$ is relatively weakly compact, and hence Asplund. Consequently, there is $m \in \mathbb{N}$ such that $L \subset mK_0 + \frac{1}{2}B(X)$. But this implies $x_n \in mK_n + \frac{1}{2}B(X_n)$ for all $n \in T$, a contradiction for $n \in T$ with $n \geq m$. Since sufficiency is straightforward, the proof is complete. $\square$

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